

ABELIAN ÉTALE COVERINGS OF GENERIC CURVES AND ORDINARINESS OF DORMANT OPERS

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ABSTRACT. In the present paper, we introduce and study a certain ordinariness of GL_n -do'pers (= dormant GL_n -opers) on a pointed proper smooth algebraic curve, which may be thought of as a generalization of the classical ordinariness of algebraic curves in positive characteristic. We consider an analogy of a type of assertion discussed and proved by S. Nakajima and M. Raynaud, i.e., ordinariness of GL_n -do'pers pulled-back via abelian coverings of the underlying curve. The main results of the present paper consist of the following two assertions: the first is that if the GL_n -do'per pulled-back via an étale covering is ordinary, then the original GL_n -do'per is ordinary; the second is that if a given GL_n -do'per is ordinary and its underlying curve is sufficiently general, then the pull-back of this GL_n -do'per via an abelian covering is ordinary whenever its Galois group has the order prime to the characteristic of the base field.

CONTENTS

Introduction	1
1. Preliminaries	4
2. GL_n -do'pers and ordinariness	12
3. Cyclic log étale coverings of a pointed projective line	20
4. GL_n -do'pers on a totally degenerate curve of genus 1	26
5. Ordinariness for a semisimple Lie algebra \mathfrak{g}	34
References	36

INTRODUCTION

0.1. In the present paper, we introduce and study a certain *ordinariness of GL_n -do'pers* (= *dormant GL_n -opers*) on a pointed proper smooth (or more

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generally, stable) algebraic curve, which may be thought of as a generalization of the classical ordinariness of algebraic curves in positive characteristic (i.e., the p -rank of its jacobian is maximal).

Recall (cf. Definition 2.1.1 (i)) that a GL_n -oper on an algebraic curve is, by definition, a triple $\mathcal{F}^\heartsuit := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n)$ consisting of a rank n vector bundle \mathcal{F} on the curve, a(n) (integrable) connection $\nabla_{\mathcal{F}}$ on \mathcal{F} , and a complete flag $\{\mathcal{F}^j\}_{j=0}^n$ on \mathcal{F} satisfying certain conditions. GL_n -opers on a family of pointed stable curves of arbitrary characteristic were defined and discussed in [23] (or, in [24]). But, various properties of opers in the case where $n = 2$ were originally discussed, under the name of indigenous bundles, in the context of the p -adic Teichmüller theory developed by S. Mochizuki (cf. [13] and [14]). Also, opers defined on a proper smooth curve over the field of complex numbers \mathbb{C} or a formal disc over \mathbb{C} were introduced and studied in the work of geometric Langlands program by A. Beilinson and V. Drinfeld (cf. [2] and [3]). Thus, opers play a central role in integrable systems and representation theory of loop algebras. If the underlying curve has characteristic $p > 0$, then it makes sense to speak of GL_n -opers with *vanishing p -curvature*, which we refer to as *dormant GL_n -opers* or *GL_n -do'pers* for short (cf. Definition 2.1.2). GL_n -do'pers and their moduli occur naturally in mathematics, as discussed in, e.g., [7], [21], and [23]. Thus, a much more understanding of them will be of use in various areas relevant to the theory of opers in positive characteristic.

0.2. As declared at the beginning of Introduction, the property on GL_n -do'pers which we want to focus on is ordinariness (cf. Definition 2.2.1). The condition of being “*ordinary*” may be described in terms of certain complexes of sheave obtained from the adjoint bundles associated with GL_n -do'pers. The precise definition of ordinariness, as well as \otimes -ordinariness, of GL_n -do'pers will be given in Definition 2.2.1. In the case where $n = 1$, the ordinariness of GL_1 -do'pers is equivalent to the classical ordinariness of their underlying curves (cf. the discussion in § 2.4, especially, Proposition 2.4.1). Hence, from this point of view, the ordinariness of GL_n -do'pers may be thought of as a higher rank analogue of the classical ordinariness.

Here, we focus our attention on the relation between the ordinariness and coverings of the curve. Let us review an assertion proved by S. Nakajima and M. Raynaud (cf. [15], [19], and [4]) as follows. Let $Y \rightarrow X$ be a Galois covering of connected proper smooth (hyperbolic) curves with Galois group G . Suppose that X is general (in the sense described in Theorem B below) and the finite group G is either abelian or a central extension of two abelian groups with $(\sharp(G), g!) = 1$. Then, it follows from [15], Theorem 2, and [19], THÉORÈME 14, that Y is ordinary. (See also [19], THÉORÈME 2 and [4] for the other assertions concerning the relation between the ordinariness and

Galois coverings.) In the present paper, we consider an analogy of such a type of assertion for ordinariness of GL_n -do'pers pulled-back via abelian coverings.

0.3. Let us describe the main results of the present paper. Let p be a prime, n a positive integer with $n < p$, (g, r) a pair of nonnegative integers with $2g - 2 + r > 0$, and k an algebraically closed field of characteristic p . Also, let $\mathfrak{X} := (X, \{\sigma_i\}_{i=1}^r)$ be a pointed proper smooth curve of type (g, r) over k , that is to say, a geometrically connected, proper, and smooth curve X of genus g over k together with r marked points $\{\sigma_i\}_{i=1}^r$ in X . Suppose that we are given a GL_n -do'per $\mathcal{F}^\heartsuit := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{i=1}^j)$ on \mathfrak{X} and an étale covering $(\mathfrak{Y}, w : Y \rightarrow X)$ of \mathfrak{X} over k (cf. Definition 1.6.1 (i) for the definition of an étale covering of a curve), where \mathfrak{Y} denotes another pointed proper smooth curve over k whose underlying curve is Y . By pulling-back the data in \mathcal{F}^\heartsuit via w , one obtains a GL_n -oper $w^*(\mathcal{F})^\heartsuit$ on \mathfrak{Y} . The goal of the present paper is to prove the following two assertions concerning the relation between the ordinariness of \mathcal{F}^\heartsuit and the ordinariness of its pull-back $w^*(\mathcal{F})^\heartsuit$.

Theorem A.

Suppose that the pull-back $w^(\mathcal{F})^\heartsuit$ is ordinary. Then \mathcal{F}^\heartsuit is ordinary.*

Theorem B.

Suppose the following conditions:

- (i) \mathcal{F}^\heartsuit is ordinary;
- (ii) (\mathfrak{Y}, w) is an abelian covering with $p \nmid \#(\mathrm{Gal}(\mathfrak{Y}/\mathfrak{X}))$ (cf. Definition 1.6.1 (i) and (ii));
- (iii) \mathfrak{X} is general in the moduli stack $\mathfrak{M}_{g,r}$ classifying pointed proper smooth curves of type (g, r) over k .

(Here, we recall that $\mathfrak{M}_{g,r}$ is irreducible (cf. [5], § 5); thus, it makes sense to speak of a “general” \mathfrak{X} , i.e., an \mathfrak{X} that determines a point of $\mathfrak{M}_{g,r}$ lying outside some fixed closed substack.) Then, the pull-back $w^(\mathcal{F})^\heartsuit$ is ordinary.*

0.4. In the final section of the present paper, we shall also consider, as a generalization of the ordinariness of GL_n -do'pers, a certain ordinariness of \mathfrak{g} -do'pers (= dormant \mathfrak{g} -oper) for a general semisimple Lie algebra \mathfrak{g} . It is a key observation (cf. Proposition 2.5.1) that the \otimes -ordinariness of GL_n -do'pers corresponds exactly the unramifiedness of the moduli stack $\mathfrak{Op}_{\mathrm{GL}_n, g, r}^{\mathrm{Zzz} \dots}$ (cf. § 2.5, (48)) of GL_n -do'pers (which is isomorphic to the moduli stack $\mathfrak{Op}_{\mathfrak{sl}_n, g, r}^{\mathrm{Zzz} \dots}$ of \mathfrak{sl}_n -do'pers) at the classifying points. Thus, by taking account of this observation, one may generalize the notion of ordinariness to the case of \mathfrak{g} -do'pers (i.e., the $\otimes_{\mathfrak{g}}$ -ordinariness defined in Definition 5.1.1) in terms of such a local property

applied to the moduli stack $\mathfrak{Op}_{\mathfrak{g},g,r}^{\text{Zzz}\dots}$ of \mathfrak{g} -do'pers. If \mathfrak{g} satisfies a certain condition, which we refer to as being *admissibly of classical type A* (cf. Definition 5.1.2), then there exists a closed immersion $\mathfrak{Op}_{\mathfrak{g},g,r}^{\text{Zzz}\dots} \rightarrow \mathfrak{Op}_{\mathfrak{sl}_m,g,r}^{\text{Zzz}\dots}$ for a sufficiently large m . This morphism allow us to relate the (\otimes) -ordinariness of GL_m -do'pers with the $\otimes_{\mathfrak{g}}$ -ordinariness (cf. Proposition 5.1.3 and Theorem 5.2.2). Moreover, for such a \mathfrak{g} , we obtain (cf. Theorem 5.1.4) an assertion which may be thought of as a generalization of Theorem B.

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1. PRELIMINARIES

First, we shall introduce our notation and review some topics involved (e.g., integrable vector bundles on log-curves, Cartier operators, and ordinariness of algebraic curves in positive characteristic). A *log scheme* means, in the present paper, a scheme equipped with a logarithmic structure in the sense of Fontaine-Illusie (cf. [10], (1.2)). For a log scheme (resp., a morphism of log schemes) indicated, say, by the notation “ X^{\log} ” (resp., “ f^{\log} ”), we shall use the notation “ X ” (resp., “ f ”) for indicating the underlying scheme (resp., the underlying morphism of schemes). Throughout the present paper, *we fix a prime p and an algebraically closed field k of characteristic $p > 0$.*

1.1. Recall from [1], Definition 4.5, the definition of a log-curve as follows.

Definition 1.1.1.

A **log-curve** is a log smooth integrable morphism $f^{\log} : X^{\log} \rightarrow S^{\log}$ of fs log schemes such that the geometric fibers of the underlying morphism $f : X \rightarrow S$ are reduced connected 1-dimensional schemes. If T^{\log} is an arbitrary log scheme, then we shall mean, by a **log-curve over T^{\log}** , a log scheme Y^{\log} over T^{\log} which is isomorphic to $X^{\log} \times_{f^{\log}, S^{\log}} T^{\log}$ for some log-curve $f^{\log} : X^{\log} \rightarrow S^{\log}$ and some morphism $T^{\log} \rightarrow S^{\log}$ of log schemes. (If $\Omega_{Y^{\log}/T^{\log}}$ denotes the sheaf of logarithmic 1-forms on Y^{\log} relative to T^{\log} , then it is a line bundle and admits a universal log derivation $d : \mathcal{O}_Y \rightarrow \Omega_{Y^{\log}/T^{\log}}$.)

Also, let us recall (families of) pointed stable curves and log-curves associated with them, as follows. Let (g, r) be a pair of nonnegative integers with $2g - 2 + r > 0$. Denote by $\overline{\mathfrak{M}}_{g,r}$ the moduli stack of r -pointed stable curves (cf. [12], Definition 1.1) over k of genus g (where we shall refer to such pointed stable curves as being **of type** (g, r)), and by $f_{\text{tau}} : \mathfrak{C}_{g,r} \rightarrow \overline{\mathfrak{M}}_{g,r}$ the tautological curve over $\overline{\mathfrak{M}}_{g,r}$, with its r marked points $\sigma_{\text{tau},1}, \dots, \sigma_{\text{tau},r} : \overline{\mathfrak{M}}_{g,r} \rightarrow \mathfrak{C}_{g,r}$. Recall (cf. [12], Corollary 2.6 and Theorem 2.7; [5], § 5) that $\overline{\mathfrak{M}}_{g,r}$ may be represented by a geometrically connected, proper, and smooth Deligne-Mumford stack over k of dimension $3g - 3 + r$. Also, recall (cf. [9], Theorem 4.5) that $\overline{\mathfrak{M}}_{g,r}$ has a natural log structure given by the divisor at infinity, where we shall denote the resulting log stack by $\overline{\mathfrak{M}}_{g,r}^{\log}$. Moreover, we obtain a log structure on $\mathfrak{C}_{g,r}$ by taking the divisor which is the union of the $\sigma_{\text{tau},i}$'s and the pull-back of the divisor at infinity of $\overline{\mathfrak{M}}_{g,r}$ (resp., the divisor defined as the pull-back of the divisor at infinity of $\overline{\mathfrak{M}}_{g,r}$); let us denote the resulting log stack by $\mathfrak{C}_{g,r}^{\log}$ (resp., $\mathfrak{C}_{g,r}^{\log'}$). $f_{\text{tau}} : \mathfrak{C}_{g,r} \rightarrow \overline{\mathfrak{M}}_{g,r}$ extends naturally to a morphism $f_{\text{tau}}^{\log} : \mathfrak{C}_{g,r}^{\log} \rightarrow \overline{\mathfrak{M}}_{g,r}^{\log}$ (resp., $f_{\text{tau}}^{\log'} : \mathfrak{C}_{g,r}^{\log'} \rightarrow \overline{\mathfrak{M}}_{g,r}^{\log}$) of log stacks. Let $\mathfrak{M}_{g,r}$ denotes the substack of $\overline{\mathfrak{M}}_{g,r}$ classifying pointed proper smooth curves; it is a dense open substack of $\overline{\mathfrak{M}}_{g,r}$ and coincides with the locus in which the log structure of $\overline{\mathfrak{M}}_{g,r}^{\log}$ becomes trivial.

Let us fix a scheme S over k and a pointed stable curve

$$(1) \quad \mathfrak{X} := (f : X \rightarrow S, \{\sigma_i : S \rightarrow X\}_{i=1}^r)$$

over S of type (g, r) , which consists of a (proper) semistable curve $f : X \rightarrow S$ over S of genus g and r marked points $\sigma_i : S \rightarrow X$ ($i = 1, \dots, r$). By pulling-back the log structures of $\overline{\mathfrak{M}}_{g,r}^{\log}$ and $\mathfrak{C}_{g,r}^{\log}$ (resp., $\mathfrak{C}_{g,r}^{\log'}$) via its classifying morphism, we obtain log structures on S and X respectively; we denote the resulting log stacks by

$$(2) \quad S^{\mathfrak{X}\text{-log}} \text{ and } X^{\mathfrak{X}\text{-log}} \text{ (resp., } X^{\mathfrak{X}\text{-log'}}).$$

If there is no fear of confusion, we shall abbreviate them to S^{\log} and X^{\log} (resp., $X^{\log'}$) respectively. The structure morphism $f : X \rightarrow S$ extends to a morphism $f^{\log} : X^{\log} \rightarrow S^{\log}$ (resp., $f^{\log'} : X^{\log'} \rightarrow S^{\log}$) of log schemes, by which the log scheme X^{\log} (resp., $X^{\log'}$) becomes a log-curve over S^{\log} . If, moreover, the underlying scheme X is smooth over S , then $S^{\mathfrak{X}\text{-log}} = S$.

Denote by $D_{\mathfrak{X}}$ the étale effective relative divisor on X relative to S defined to be the union of the image of the marked points σ_i ($i = 1, \dots, r$). If $\omega_{X/S}$ denotes the dualizing sheaf of X over S , then $\omega_{X/S}$ is isomorphic to $\Omega_{X^{\log}/S^{\log}}(-D_{\mathfrak{X}})$ ($= \Omega_{X^{\log'}/S^{\log}}$).

1.2. Let S^{\log} be a log scheme and $f^{\log} : X^{\log} \rightarrow S^{\log}$ a log-curve over S^{\log} , and \mathcal{F} a rank n (≥ 0) vector bundle (i.e., a locally free coherent sheaf of finite rank) on X . By an **S -log connection** on \mathcal{F} , we mean an $f^{-1}(\mathcal{O}_S)$ -linear morphism

$$(3) \quad \nabla : \mathcal{F} \rightarrow \Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F}$$

satisfying the condition that

$$(4) \quad \nabla(a \cdot m) = d(a) \otimes m + a \cdot \nabla(m)$$

for any local sections $a \in \mathcal{O}_X$, $m \in \mathcal{F}$.

An **integrable vector bundle** on X^{\log}/S^{\log} (of rank n) is a pair $(\mathcal{F}, \nabla_{\mathcal{F}})$ consisting of a vector bundle \mathcal{F} on X (of rank n) and an S -log connection $\nabla_{\mathcal{F}}$ on \mathcal{F} . (Note that since $\Omega_{X^{\log}/S^{\log}}$ is a line bundle, any S -log connection on a vector bundle is automatically integrable, i.e., has vanishing curvature.) If, moreover, \mathcal{F} is of rank 1, then we shall refer to $(\mathcal{F}, \nabla_{\mathcal{F}})$ as an **integrable line bundle** on X^{\log}/S^{\log} . For simplicity, we shall refer to an integral vector bundle on $X^{\mathfrak{X}\text{-log}}/S^{\mathfrak{X}\text{-log}}$ (where \mathfrak{X} denotes a pointed stable curve whose underlying curve coincides with X) as an **integrable vector bundle on \mathfrak{X}** .

Let $(\mathcal{F}, \nabla_{\mathcal{F}})$ and $(\mathcal{G}, \nabla_{\mathcal{G}})$ be integrable vector bundles on X^{\log}/S^{\log} . We shall write $\nabla_{\mathcal{F}} \otimes \nabla_{\mathcal{G}}$ (by abuse of notation) for the S -log connection on the tensor product $\mathcal{F} \otimes \mathcal{G}$ determined by

$$(5) \quad (\nabla_{\mathcal{F}} \otimes \nabla_{\mathcal{G}})(a \otimes b) = \nabla_{\mathcal{F}}(a) \otimes b + a \otimes \nabla_{\mathcal{G}}(b).$$

for any local sections $a \in \mathcal{F}$ and $b \in \mathcal{G}$. Also, we define an **isomorphism of integrable vector bundles** from $(\mathcal{F}, \nabla_{\mathcal{F}})$ to $(\mathcal{G}, \nabla_{\mathcal{G}})$ is an isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{G}$ of vector bundles that is compatible with the respective connections $\nabla_{\mathcal{F}}$ and $\nabla_{\mathcal{G}}$.

Next, suppose that we are given an integrable vector bundle $(\mathcal{F}, \nabla_{\mathcal{F}})$ on X^{\log}/S^{\log} and a commutative square diagram

$$(6) \quad \begin{array}{ccc} Y^{\log} & \xrightarrow{w^{\log}} & X^{\log} \\ \downarrow & & \downarrow \\ T^{\log} & \longrightarrow & S^{\log} \end{array}$$

of log schemes, where T^{\log} denotes a log scheme over k and Y^{\log} denotes a log-curve over T^{\log} . (Hence, we have a morphism $w^{\sharp} : w^{-1}(\Omega_{X^{\log}/S^{\log}}) \rightarrow \Omega_{Y^{\log}/T^{\log}}$ induced by w .) One may obtain a T -log connection

$$(7) \quad \nabla_{w^*(\mathcal{F})} : w^*(\mathcal{F}) \rightarrow \Omega_{Y^{\log}/T^{\log}} \otimes w^*(\mathcal{F})$$

on the pull-back $w^*(\mathcal{F})$ ($= \mathcal{O}_Y \otimes_{w^{-1}(\mathcal{O}_X)} w^{-1}(\mathcal{F})$) given by assigning

$$(8) \quad a \otimes v \mapsto d(a) \otimes v + a \cdot (w^{\sharp} \otimes \text{id}_{w^{-1}(\mathcal{F})})(w^{-1}(\nabla_{\mathcal{F}})(v))$$

for any local sections $a \in \mathcal{O}_Y$, $v \in w^{-1}(\mathcal{F})$. Thus, we have an integrable vector bundle

$$(9) \quad (w^*(\mathcal{F}), \nabla_{w^*(\mathcal{F})})$$

on Y^{\log}/T^{\log} .

1.3. In the rest of this section, we fix a scheme S over k and a pointed stable curve $\mathfrak{X} := (f : X \rightarrow S, \{\sigma_i\}_{i=1}^r)$ over S of type (g, r) (hence, we have a log-curve $X^{\log}/S^{\log} := X^{\mathfrak{X}\text{-log}}/S^{\mathfrak{X}\text{-log}}$). Denote by $F_S : S \rightarrow S$ (resp., $F_X : X \rightarrow X$) the absolute Frobenius morphism of S (resp., X). The *Frobenius twist of X over S* is, by definition, the base-change $X_S^{(1)} := X \times_{S, F_S} S$ of $f : X \rightarrow S$ via $F_S : S \rightarrow S$. Denote by $f^{(1)} : X_S^{(1)} \rightarrow S$ the structure morphism of the Frobenius twist of X over S . The *relative Frobenius morphism of X over S* is the unique morphism $F_{X/S} : X \rightarrow X_S^{(1)}$ over S that fits into a commutative diagram of the form

$$(10) \quad \begin{array}{ccccc} X & & & & \\ & \searrow^{F_X} & & \searrow & \\ & & X_S^{(1)} & \xrightarrow{\text{id}_X \times F_S} & X \\ & \searrow^{F_{X/S}} & \downarrow f^{(1)} & \square & \downarrow f \\ & & S & \xrightarrow{F_S} & S \end{array}$$

If we write

$$(11) \quad \sigma_i^{(1)} := (\sigma_i \circ F_S, \text{id}_S) : S \rightarrow X \times_{S, F_S} S (= X_S^{(1)})$$

(for each $i \in \{1, \dots, r\}$), then the collection of data

$$(12) \quad \mathfrak{X}^{(1)} := (f^{(1)} : X_S^{(1)} \rightarrow S, \{\sigma_i^{(1)}\}_{i=1}^r)$$

forms a pointed stable curve over S of type (g, r) . In particular, we obtain canonically a log structure on $X_S^{(1)}$.

If $\mathfrak{Y} := (Y/S, \{\sigma'_i\}_{i=1}^{r'})$ is a pointed stable curve over S (of type (g', r') for some pair (g', r')) and $w : Y \rightarrow X$ is a morphism over S , then we shall denote by $w^{(1)} : Y_S^{(1)} \rightarrow X_S^{(1)}$ the base-change of w via $F_S : \text{Spec}(S) \rightarrow \text{Spec}(S)$.

1.4. In this subsection, we shall recall the Cartier operator associated with an integrable vector bundle. Let $(\mathcal{F}, \nabla_{\mathcal{F}})$ be an integrable vector bundle on \mathfrak{X} with *vanishing p -curvature* (cf. [23], Definition 3.2.1, for the definition of p -curvature). Although $\nabla_{\mathcal{F}}$ is not \mathcal{O}_X -linear, but it may be thought, via the underlying homeomorphism of $F_{X/S}$, of as an $\mathcal{O}_{X_S^{(1)}}$ -linear morphism

$$(13) \quad F_{X/S*}(\nabla_{\mathcal{F}}) : F_{X/S*}(\mathcal{F}) \rightarrow F_{X/S*}(\Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F}).$$

In particular, both $F_{X/S*}(\text{Ker}(\nabla_{\mathcal{F}})) (= \text{Ker}(F_{X/S*}(\nabla_{\mathcal{F}})))$ and $F_{X/S*}(\text{Coker}(\nabla_{\mathcal{F}})) (= \text{Coker}(F_{X/S*}(\nabla_{\mathcal{F}})))$ may be thought of as $\mathcal{O}_{X_S^{(1)}}$ -modules.

The *Cartier operator* (cf. [16], Proposition 1.2.4) associated with $(\mathcal{F}, \nabla_{\mathcal{F}})$ is, by definition, a unique $\mathcal{O}_{X_S^{(1)}}$ -linear morphism

$$(14) \quad C^{(\mathcal{F}, \nabla_{\mathcal{F}})} : F_{X/S*}(\Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F}) \rightarrow \Omega_{X_S^{(1)\log}/S^{\log}} \otimes F_{X/S*}(\mathcal{F})$$

satisfying the following condition: for any locally defined logarithmic derivation $\partial \in \mathcal{T}_{X^{\log}/S^{\log}}$ and any local section $a \in \Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F}$, the equality

$$(15) \quad \langle C^{(\mathcal{F}, \nabla_{\mathcal{F}})}(a), (\text{id}_X \times F_S)^*(\partial) \rangle = \langle a, \partial^{[p]} \rangle - \nabla_{\mathcal{F}}(\partial)^{\circ(p-1)}(\langle a, \partial \rangle)$$

is satisfied, where $\nabla_{\mathcal{F}}(\partial)^{\circ(p-1)}$ denotes the $(p-1)$ -st iterate of the endomorphism $\nabla_{\mathcal{F}}(\partial)$ of \mathcal{F} , and $\langle -, - \rangle$ in the both sides denote the pairings induced by the natural pairing $\Omega_{X^{\log}/S^{\log}} \times \mathcal{T}_{X^{\log}/S^{\log}} \rightarrow \mathcal{O}_X$. Moreover, since $(\mathcal{F}, \nabla_{\mathcal{F}})$ has vanishing p -curvature, there exists (cf. [16], the discussion following Proposition 1.2.4) a commutative square

$$(16) \quad \begin{array}{ccc} F_{X/S*}(\Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F}) & \xrightarrow{C^{(\mathcal{F}, \nabla_{\mathcal{F}})}} & \Omega_{X_S^{(1)\log}/S^{\log}} \otimes F_{X/S*}(\mathcal{F}) \\ \downarrow & & \uparrow \\ F_{X/S*}(\text{Coker}(\nabla_{\mathcal{F}})) & \xrightarrow{\overline{C}^{(\mathcal{F}, \nabla_{\mathcal{F}})}} & \Omega_{X_S^{(1)\log}/S^{\log}} \otimes F_{X/S*}(\text{Ker}(\nabla_{\mathcal{F}})), \end{array}$$

where the left-hand and right-hand vertical arrows denote the $\mathcal{O}_{X_S^{(1)}}$ -linear morphisms induced by the quotient $\Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F} \twoheadrightarrow \text{Coker}(\nabla_{\mathcal{F}})$ and the inclusion $\text{Ker}(\nabla_{\mathcal{F}}) \hookrightarrow \mathcal{F}$ respectively. Finally, it follows from [17], Theorem 3.1.1, that the morphism $\overline{C}^{(\mathcal{F}, \nabla_{\mathcal{F}})}$ (i.e., the lower horizontal arrow in (16)) is an isomorphism. (Here, we note that since X^{\log}/S^{\log} is of Cartier type in the sense of [10], Definition (4.8), the notion of the exact relative Frobenius map in the statement of [16] coincides with $F_{X/S}$). Denote by $d' : \mathcal{O}_X \rightarrow \omega_{X/S} (= \Omega_{X^{\log'}/S^{\log}})$ the universal log derivation of $X^{\log'}/S^{\log}$. By composing $\overline{C}^{(\mathcal{O}_X, d)}$ (resp., $\overline{C}^{(\mathcal{O}_X, d')}$) with the quotient $\Omega_{X^{\log}/S^{\log}} \rightarrow \text{Coker}(d)$ (resp., $\omega_{X/S} \twoheadrightarrow \text{Coker}(d')$), we obtain an $\mathcal{O}_{X_S^{(1)}}$ -linear morphism

$$(17) \quad \begin{aligned} C_{X/S} : F_{X/S*}(\Omega_{X^{\log}/S^{\log}}) &\rightarrow \Omega_{X_S^{(1)\log}/S^{\log}} \\ (\text{resp.}, C'_{X/S} : F_{X/S*}(\omega_{X/S}) &\rightarrow \omega_{X_S^{(1)}/S}). \end{aligned}$$

1.5. Let us recall the ordinariness of algebraic curves, as follows.

Definition 1.5.1.

We shall say that \mathfrak{X} is **ordinary** if the \mathcal{O}_S -linear morphism

$$(18) \quad \mathbb{R}^1 f_*^{(1)}(F_{X/S}^*) : \mathbb{R}^1 f_*^{(1)}(\mathcal{O}_{X_S^{(1)}}) \rightarrow \mathbb{R}^1 f_*(\mathcal{O}_X) (= \mathbb{R}^1 f_*^{(1)}(F_{X/S*}(\mathcal{O}_X)))$$

induced from $F_{X/S}^* : \mathcal{O}_{X_S^{(1)}} \rightarrow F_{X/S*}(\mathcal{O}_X)$ is an isomorphism (cf. e.g., [4], Definition 1.2, for the definition of the ordinariness of semistable curves over an algebraically closed field).

It is well-known that the dual of $\mathbb{R}^1 f_*^{(1)}(F_{X/S}^*)$ coincides with

$$(19) \quad f_*^{(1)}(C'_{X/S}) : (f_*^{(1)}(F_{X/S*}(\omega_{X/S})) =) f_*(\omega_{X/S}) \rightarrow f_*^{(1)}(\omega_{X_S^{(1)}/S})$$

via Grothendieck-Serre duality. In particular, \mathfrak{X} is ordinary if and only if $f_*^{(1)}(C'_{X/S})$ is an isomorphism. Recall that the locus of $\overline{\mathfrak{M}}_{g,r}$ classifying ordinary pointed stable curves is open and dense.

Proposition 1.5.2.

\mathfrak{X} is ordinary if and only if $f_*^{(1)}(C_{X/S}) : f_*(\Omega_{X^{\log}/S^{\log}}) \rightarrow f_*(\Omega_{X_S^{(1)\log}/S^{\log}})$ is an isomorphism.

Proof. It suffices to consider the case where $r > 0$. Consider the morphism of short exact sequences

$$(20) \quad \begin{array}{ccccccc} 0 & \longrightarrow & F_*(\omega_{X/S}) & \xrightarrow{\text{incl}} & F_*(\Omega_{X^{\log}/S^{\log}}) & \longrightarrow & \bigoplus_{i=1}^r F_*(\sigma_{i*}^{(1)}(\mathcal{O}_S)) \longrightarrow 0 \\ & & \downarrow C'_{X/S} & & \downarrow C_{X/S} & & \downarrow \bigoplus \text{id} \\ 0 & \longrightarrow & \omega_{X_S^{(1)}/S} & \xrightarrow{\text{incl}} & \Omega_{X_S^{(1)\log}/S^{\log}} & \longrightarrow & \bigoplus_{i=1}^r \sigma_{i*}(\mathcal{O}_S) \longrightarrow 0. \end{array}$$

By taking the higher direct images via $f^{(1)}$, (20) gives rise to a morphism of complexes

$$(21) \quad \begin{array}{ccccccccccc} 0 & \longrightarrow & f_*(\omega_{X/S}) & \longrightarrow & f_*(\Omega_{X^{\log}/S^{\log}}) & \longrightarrow & \mathcal{O}_S^{\oplus r} & \longrightarrow & \mathbb{R}^1 f_*(\omega_{X/S}) & \longrightarrow & 0 \\ \downarrow \text{id} & & \downarrow f_*^{(1)}(C'_{X/S}) & & \downarrow f_*^{(1)}(C_{X/S}) & & \downarrow \text{id} & & \downarrow \mathbb{R}^1 f_*^{(1)}(C'_{X/S}) & & \downarrow \text{id} \\ 0 & \longrightarrow & f_*^{(1)}(\omega_{X_S^{(1)}/S}) & \longrightarrow & f_*^{(1)}(\Omega_{X_S^{(1)\log}/S^{\log}}) & \longrightarrow & \mathcal{O}_S^{\oplus r} & \longrightarrow & \mathbb{R}^1 f_*^{(1)}(\omega_{X_S^{(1)}/S}) & \longrightarrow & 0. \end{array}$$

But, the morphism

$$(22) \quad \mathbb{R}^1 f_*^{(1)}(C') : \mathbb{R}^1 f_*(\omega_{X/S}) \rightarrow \mathbb{R}^1 f_*^{(1)}(\omega_{X_S^{(1)}/S})$$

(i.e., the second vertical arrow from the right in (21)) is an isomorphism since it coincides, by Grothendieck-Serre duality, with the dual of the identity morphism of \mathcal{O}_S . Hence, by the five lemma, $f_*^{(1)}(C_{X/S})$ is an isomorphism if and only if $f_*^{(1)}(C'_{X/S})$ is an isomorphism. This completes the proof of Proposition 1.5.2. \square

1.6. We shall give the definition of an étale covering between pointed proper smooth curves, appeared in the statement of the main results.

Definition 1.6.1.

(i) An **étale covering of \mathfrak{X}** is a triple

$$(23) \quad (\mathfrak{Y}, w, v),$$

where

- \mathfrak{Y} denotes a pointed stable curve $\mathfrak{Y} := (Y/T, \{\sigma'_i\}_{i=1}^{r'})$ over a k -scheme T (of type (g', r') for some pair of nonnegative integers (g', r') with $2g' - 2 + r' > 0$);

• w and v denote morphisms of k -schemes $w : Y \rightarrow X, v : T \rightarrow S$, satisfying the following conditions:

(i-1) the square diagram

$$(24) \quad \begin{array}{ccc} Y & \xrightarrow{w} & X \\ \downarrow & & \downarrow \\ T & \xrightarrow{v} & S \end{array}$$

is commutative, where the two vertical arrows denote the structure morphisms of pointed stable curves;

(i-2) the morphism $w_v : Y \rightarrow X \times_S T$ over T induced by (24) is finite and étale satisfying that $(\text{pr} \circ w_v)^{-1}(D_{\mathfrak{X}}) = D_{\mathfrak{Y}}$, where pr denotes the projection $X \times_S T \rightarrow X$.

(Hence, the natural morphism $w^*(\Omega_{X^{\log}/S^{\log}}) \rightarrow \Omega_{Y^{\log}/T^{\log}}$ is an isomorphism.) An **étale covering of \mathfrak{X} over S** is an étale covering (\mathfrak{Y}, w, v) of \mathfrak{X} such that \mathfrak{Y} is a pointed stable curve over S and $v = \text{id}_S$. We shall abbreviate $(\mathfrak{Y}, w, \text{id}_S)$ to (\mathfrak{Y}, w) for simplicity.

(ii) We shall say that an étale covering (\mathfrak{Y}, w, v) of \mathfrak{X} (over S) is **Galois** (resp., **abelian**) if w_v is a Galois covering (resp., an abelian covering). In this situation, we shall write

$$(25) \quad \text{Gal}(\mathfrak{Y}/\mathfrak{X}) := \text{Gal}(Y/X \times_S T)$$

and refer to it as the **Galois group** of (\mathfrak{Y}, w, v) over \mathfrak{X} .

(iii) Let (\mathfrak{Y}, w_Y) and (\mathfrak{Z}, w_Z) be an étale coverings of \mathfrak{X} over S (where we denote by Y and Z the underlying semistable curves respectively). We shall say that (\mathfrak{Y}, w_Y) is **isomorphic to (\mathfrak{Z}, w_Z)** if there exists an isomorphism $h : Y \xrightarrow{\sim} Z$ over X such that $h^{-1}(D_{\mathfrak{Z}}) = D_{\mathfrak{Y}}$. (Thus, the isomorphism class of an étale covering (Y, w, id_S) depends only on the étale morphism $w : Y \rightarrow X$, i.e., not on the marked points.)

Finally, we shall consider a relation between the sheaf of horizontal sections of an integrable vector bundle and its pull-back via an étale covering.

Proposition 1.6.2.

Let $\mathfrak{Y} := (Y/S, \{\sigma'_i\}_{i=1}^{r'})$ be an étale covering of \mathfrak{X} and denote by $w : Y \rightarrow X$ the structure morphism of Y over X . Let us consider the commutative square diagram of k -schemes

$$(26) \quad \begin{array}{ccc} Y & \xrightarrow{w} & X \\ F_{Y/S} \downarrow & & \downarrow F_{X/S} \\ Y_S^{(1)} & \xrightarrow{w^{(1)}} & X_S^{(1)} \end{array}$$

Then, for each \mathcal{O}_X -module \mathcal{F} , the morphism

$$(27) \quad w^{(1)*}(F_{X/S*}(\mathcal{F})) \rightarrow F_{Y/S*}(w^*(\mathcal{F}))$$

of $\mathcal{O}_{Y_S^{(1)}}$ -module induced, via adjunction, from this square diagram is an isomorphism and functorial in \mathcal{F} . Moreover, if $(\mathcal{F}, \nabla_{\mathcal{F}})$ is an integrable vector bundle on \mathfrak{X} , then the isomorphism (27) restricts to an isomorphism

$$(28) \quad w^{(1)*}(F_{X/S*}(\text{Ker}(\nabla_{\mathcal{F}}))) \xrightarrow{\sim} F_{Y/S*}(\text{Ker}(\nabla_{w^*(\mathcal{F})}))$$

of $\mathcal{O}_{Y_S^{(1)}}$ -modules.

Proof. The former assertion may be immediately verified. Indeed, the étaleness of w implies that the diagram (26) is cartesian, and hence, (27) is an isomorphism.

Next, let us consider the latter assertion. The isomorphism $w^*(\Omega_{X^{\log}/S^{\log}}) \xrightarrow{\sim} \Omega_{Y^{\log}/S^{\log}}$ (due to the étaleness of w) and the isomorphism (27) of the case where \mathcal{F} is replaced with $\Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F}$ yield a composite isomorphism

$$(29) \quad \begin{aligned} w^{(1)*}(F_{X/S*}(\Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F})) &\xrightarrow{\sim} F_{Y/S*}(w^*(\Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F})) \\ &\xrightarrow{\sim} F_{Y/S*}(\Omega_{Y^{\log}/S^{\log}} \otimes w^*(\mathcal{F})). \end{aligned}$$

One verifies immediately that the square diagram

$$(30) \quad \begin{array}{ccc} w^{(1)*}(F_{X/S*}(\mathcal{F})) & \xrightarrow{w^{(1)*}(F_{X/S*}(\nabla_{\mathcal{F}}))} & w^{(1)*}(F_{X/S*}(\Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F})) \\ (27) \downarrow \wr & & \wr \downarrow (29) \\ F_{Y/S*}(w^*(\mathcal{F})) & \xrightarrow{F_{Y/S*}(\nabla_{w^*(\mathcal{F})})} & F_{Y/S*}(\Omega_{Y^{\log}/S^{\log}} \otimes w^*(\mathcal{F})) \end{array}$$

is commutative. In particular, by taking the kernels of the upper and lower horizontal arrows, we obtain the isomorphism (28), as desired. \square

2. GL_n -DO'PERS AND ORDINARINESS

In this section, we shall recall the definition of a GL_n -oper, as well as a GL_n -do'per (cf. Definition 2.1.1 and Definition 2.1.2), and then, introduce the notion of $(*)$ -ordinarinesses of GL_n -do'pers (cf. Definition 2.2.1). Moreover, we shall discuss several properties concerning the ordinariness. At the end of this section, the proof of Theorem A, being one of our main results, will be given. In the following, *we fix a positive integer n with $n < p$.*

2.1. We first recall (cf. [23], Definition 4.2.1 and [24], Definition 4.2.1) the definition of a GL_n -oper as follows. Let S^{\log} be a log scheme over k and $f^{\log} : X^{\log} \rightarrow S^{\log}$ be a log-curve over S^{\log} .

Definition 2.1.1.

(i) A GL_n -**oper** on X^{\log}/S^{\log} is a collection of data

$$(31) \quad \mathcal{F}^{\heartsuit} := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n),$$

where

- \mathcal{F} is a vector bundle on X of rank n ;
- $\nabla_{\mathcal{F}}$ is an S -log connection $\mathcal{F} \rightarrow \Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F}$ on \mathcal{F} ;
- $\{\mathcal{F}^j\}_{j=0}^n$ is a decreasing filtration:

$$(32) \quad 0 = \mathcal{F}^n \subseteq \mathcal{F}^{n-1} \subseteq \dots \subseteq \mathcal{F}^0 = \mathcal{F}$$

on \mathcal{F} by vector bundles on X ,

satisfying the following conditions:

- (i-1) the subquotients $\mathcal{F}^j/\mathcal{F}^{j+1}$ ($0 \leq j \leq n-1$) are line bundles;
- (i-2) $\nabla_{\mathcal{F}}(\mathcal{F}^j) \subseteq \Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F}^{j-1}$ ($1 \leq j \leq n-1$);
- (i-3) the \mathcal{O}_X -linear morphisms

$$(33) \quad \mathfrak{ts}_{\mathcal{F}^{\heartsuit}}^j : \mathcal{F}^j/\mathcal{F}^{j+1} \rightarrow \Omega_{X^{\log}/S^{\log}} \otimes (\mathcal{F}^{j-1}/\mathcal{F}^j)$$

($1 \leq j \leq n-1$) defined by assigning $\bar{a} \mapsto \overline{\nabla_{\mathcal{F}}(a)}$ for any local section $a \in \mathcal{F}^j$ (where $\overline{(-)}$'s denote the images in the respective quotients), which are well-defined by virtue of the condition (i-2), are isomorphisms.

For simplicity, we shall refer to a GL_n -oper on $X^{\mathfrak{X}\text{-log}}/S^{\mathfrak{X}\text{-log}}$ (where \mathfrak{X} denotes a pointed stable curve whose underlying curve coincides with X) as a GL_n -**oper on \mathfrak{X}** .

- (ii) Let $\mathcal{F}^{\heartsuit} := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n)$ and $\mathcal{G}^{\heartsuit} := (\mathcal{G}, \nabla_{\mathcal{G}}, \{\mathcal{G}^j\}_{j=0}^n)$ be GL_n -opers on X^{\log}/S^{\log} . An **isomorphism of GL_n -opers** from \mathcal{F}^{\heartsuit} to \mathcal{G}^{\heartsuit} is an isomorphism of integrable vector bundles from $(\mathcal{F}, \nabla_{\mathcal{F}})$ to $(\mathcal{G}, \nabla_{\mathcal{G}})$ that preserves the respective filtrations $\{\mathcal{F}^j\}_{j=0}^n$ and $\{\mathcal{G}^j\}_{j=0}^n$.

Definition 2.1.2.

We shall say that a GL_n -oper $\mathcal{F}^\heartsuit := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n)$ is **dormant** if the p -curvature of $\nabla_{\mathcal{F}}$ vanishes identically on X (cf. e.g., [23], Definition 3.2.1, for the definition of p -curvature). For convenience, we shall abbreviate a “dormant GL_n -oper” to a “ GL_n -do’per”.

Let us consider the pull-back of a GL_n -oper. Let T^{\log} be a log scheme over k and Y^{\log} a log-curve over T^{\log} . Suppose that we are given a GL_n -oper $\mathcal{F}^\heartsuit := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\sigma_i\}_{i=1}^r)$ on X^{\log}/S^{\log} and a pair (w^{\log}, v^{\log}) consisting of morphisms $w^{\log} : Y^{\log} \rightarrow X^{\log}$, $v^{\log} : T^{\log} \rightarrow S^{\log}$ which make the square diagram

$$(34) \quad \begin{array}{ccc} Y^{\log} & \xrightarrow{w^{\log}} & X^{\log} \\ \downarrow & & \downarrow \\ T^{\log} & \xrightarrow{v^{\log}} & S^{\log} \end{array}$$

commute and such that the induced morphism $Y^{\log} \rightarrow X^{\log} \times_{S^{\log}} T^{\log}$ is log étale. Write $w^*(\mathcal{F})^j := w^*(\mathcal{F}^j)$ ($j = 0, \dots, n$). Then, since w^{\log} induces an isomorphism $w^*(\Omega_{X^{\log}/S^{\log}}) \xrightarrow{\sim} \Omega_{Y^{\log}/T^{\log}}$, the collection of data

$$(35) \quad w^*(\mathcal{F})^\heartsuit := (w^*(\mathcal{F}), \nabla_{w^*(\mathcal{F})}, \{w^*(\mathcal{F}^j)\}_{j=0}^n)$$

forms a GL_n -oper on Y^{\log}/T^{\log} . If, moreover, \mathcal{F}^\heartsuit is dormant, then $w^*(\mathcal{F})^\heartsuit$ is dormant.

Definition 2.1.3.

We shall refer to the GL_n -oper $w^*(\mathcal{F})^\heartsuit$ on Y^{\log}/T^{\log} defined above as the **pull-back of \mathcal{F}^\heartsuit via (w^{\log}, v^{\log})** .

2.2. Now, we consider certain ordinarinesses of GL_n -do’pers on pointed stable curves. Let S be a scheme over k , $\mathfrak{X} := (f : X \rightarrow S, \{\sigma_i\}_{i=1}^r)$ a pointed stable curve over S , and $\mathcal{F}^\heartsuit := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n)$ a GL_n -do’per on \mathfrak{X} . Consider an \mathcal{O}_X -module

$$(36) \quad \mathcal{E}nd^{\oplus}(\mathcal{F}) \ (\subseteq \mathcal{E}nd(\mathcal{F}))$$

defined to be the sheaf of \mathcal{O}_X -linear endomorphisms of \mathcal{F} with vanishing trace. In the following,

the symbol \square denotes either the presence or absence of \oplus .

The filtration $\{\mathcal{F}^j\}_{j=0}^n$ carries a decreasing filtration $\{\mathcal{E}nd^{\square}(\mathcal{F})^j\}_{j \in \mathbb{Z}}$ on $\mathcal{E}nd^{\square}(\mathcal{F})$ given by

$$(37) \quad \mathcal{E}nd^{\square}(\mathcal{F})^j := \{f \in \mathcal{E}nd^{\square}(\mathcal{F}) \mid f(\mathcal{F}^l) \subseteq \mathcal{F}^{l+j} \text{ for any } l\}.$$

Let

$$(38) \quad \nabla_{\mathcal{F}}^{\text{ad}\square} : \mathcal{E}nd^{\square}(\mathcal{F}) \rightarrow \Omega_{X^{\log}/S^{\log}} \otimes \mathcal{E}nd^{\square}(\mathcal{F})$$

denotes the S -log connection on $\mathcal{E}nd^{\square}(\mathcal{F})$ induced naturally by the S -log connection $\nabla_{\mathcal{F}}$. By the definition of a GL_n -oper, the restriction of $\nabla_{\mathcal{F}}^{\text{ad}\square}$ to $\mathcal{E}nd^{\square}(\mathcal{F})^j$ (for each $j \in \mathbb{Z}$) determines an $f^{-1}(\mathcal{O}_S)$ -linear morphism

$$(39) \quad \nabla_{\mathcal{F}}^{\text{ad}\square, j} : \mathcal{E}nd^{\square}(\mathcal{F})^j \rightarrow \Omega_{X^{\log}/S^{\log}} \otimes \mathcal{E}nd^{\square}(\mathcal{F})^{j-1}.$$

Since the composite

$$(40) \quad \begin{aligned} f_*(\mathcal{E}nd^{\square}(\mathcal{F})^j) &\xrightarrow{f_*(\nabla_{\mathcal{F}}^{\text{ad}\square, j})} f_*(\Omega_{X^{\log}/S^{\log}} \otimes \mathcal{E}nd^{\square}(\mathcal{F})^{j-1}) \\ &\hookrightarrow f_*(\Omega_{X^{\log}/S^{\log}} \otimes \mathcal{E}nd^{\square}(\mathcal{F})) \\ &\rightarrow f_*(\text{Coker}(\nabla_{\mathcal{F}}^{\text{ad}\square})) \end{aligned}$$

becomes the zero map, the composite of the second and third arrows in (40) gives rise to an \mathcal{O}_S -linear morphism

$$(41) \quad \Theta_{\mathcal{F}^{\heartsuit}}^{\square, j} : \text{Coker}(f_*(\nabla_{\mathcal{F}}^{\text{ad}\square, j})) \rightarrow f_*(\text{Coker}(\nabla_{\mathcal{F}}^{\text{ad}\square})).$$

Definition 2.2.1.

We shall say that a GL_n -do'per \mathcal{F}^{\heartsuit} is **ordinary** (resp., **\otimes -ordinary**) if $\Theta_{\mathcal{F}^{\heartsuit}}^1$ (resp., $\Theta_{\mathcal{F}^{\heartsuit}}^{\otimes, 1}$) is injective.

2.3. Let \mathfrak{X} and \mathcal{F}^{\heartsuit} be as above and $(\mathcal{L}, \nabla_{\mathcal{L}})$ an integrable line bundle on \mathfrak{X} . Then, the collection of data

$$(42) \quad \mathcal{F}_{(\mathcal{L}, \nabla_{\mathcal{L}})}^{\heartsuit} := (\mathcal{F} \otimes \mathcal{L}, \nabla_{\mathcal{F}} \otimes \nabla_{\mathcal{L}}, \{\mathcal{F}^j \otimes \mathcal{L}\}_{j=0}^n)$$

(cf. (5) for the definition of $\nabla_{\mathcal{F}} \otimes \nabla_{\mathcal{L}}$) forms a GL_n -oper on \mathfrak{X} . Moreover, one verifies that the p -curvature of $\nabla_{\mathcal{L}}$ vanishes identically on X if and only if $\mathcal{F}_{(\mathcal{L}, \nabla_{\mathcal{L}})}^{\heartsuit}$ is dormant.

Then, we shall define an equivalence relation in the set of GL_n -do'pers on \mathfrak{X} , as follows.

Definition 2.3.1.

Let \mathcal{F}^{\heartsuit} and \mathcal{G}^{\heartsuit} be GL_n -do'pers on \mathfrak{X} . We shall say that \mathcal{F}^{\heartsuit} is **equivalent to** \mathcal{G}^{\heartsuit} if \mathcal{F}^{\heartsuit} is isomorphic to $\mathcal{G}_{(\mathcal{L}, \nabla_{\mathcal{L}})}^{\heartsuit}$ for some integrable line bundle $(\mathcal{L}, \nabla_{\mathcal{L}})$ on \mathfrak{X} (with vanishing p -curvature).

Remark 2.3.1.1.

Let $(\mathcal{L}, \nabla_{\mathcal{L}})$ be an integrable line bundle on \mathfrak{X} . $\mathcal{E}nd^{\square}(\mathcal{F})$ is canonically identified with $\mathcal{E}nd^{\square}(\mathcal{F} \otimes \mathcal{L})$ and the S -log connection $\nabla_{\mathcal{F}}^{\text{ad}\square}$ coincides with $\nabla_{\mathcal{F} \otimes \mathcal{L}}^{\text{ad}\square}$ via this identification. It follows that \mathcal{F}^{\heartsuit} is \square -ordinary if and only if $\mathcal{F}_{(\mathcal{L}, \nabla_{\mathcal{L}})}^{\heartsuit}$ is \square -ordinary. In particular, *the \square -ordinariness of GL_n -do'pers depends only on their equivalence classes.*

2.4. We shall consider the case where $n = 1$. Any integrable line bundle $(\mathcal{L}, \nabla_{\mathcal{L}})$ may be equipped with a trivial 1-step filtration $\{\mathcal{L}^j\}_{j=0}^1$ given by $\mathcal{L}^0 := \mathcal{L}$ and $\mathcal{L}^1 := 0$. This filtration allows us to consider $(\mathcal{L}, \nabla_{\mathcal{L}})$ as a GL_1 -oper $\mathcal{L}^{\heartsuit} := (\mathcal{L}, \nabla_{\mathcal{L}}, \{\mathcal{L}^j\}_{j=0}^1)$ on \mathfrak{X} . By this way, *we may identify any integrable line bundle (resp., any integrable line bundle with vanishing p -curvature) with a GL_1 -oper (resp., a GL_1 -do'per).* In particular, one obtains the **trivial GL_1 -do'per**

$$(43) \quad \mathcal{O}_X^{\heartsuit} := (\mathcal{O}_X, d, \{\mathcal{O}_X^j\}_{j=0}^1)$$

on \mathfrak{X} . It is clear that any two GL_1 -do'pers are equivalent. In other words, *any GL_1 -do'per is equivalent to the trivial GL_1 -do'per $\mathcal{O}_X^{\heartsuit}$.*

Next, consider the \square -ordinariness. According to the above discussion and the discussion in Remark 2.3.1.1, an arbitrary GL_1 -do'per is \square -ordinary if and only if $\mathcal{O}_X^{\heartsuit}$ is \square -ordinary. Thus, it suffices to consider the case of the trivial GL_1 -do'per.

On the one hand, if $\square = \otimes$, then $\mathcal{O}_X^{\heartsuit}$ is necessarily \otimes -ordinary since $\mathcal{E}nd^{\otimes}(\mathcal{O}_X) = 0$. On the other hand, suppose that \square denotes the absence of \otimes . We have the equality $\nabla_{\mathcal{O}_X}^{\text{ad}} = d$, and $\nabla_{\mathcal{O}_X}^{\text{ad},1}$ coincides with the zero map $0 \rightarrow \Omega_{X/\log/S}$. Thus, $\Theta_{\mathcal{O}_X^{\heartsuit}}^1$ coincides with $f_*^{(1)}(C_{X/S})$ (cf. (17)). It follows from Proposition 1.5.2 that $\mathcal{O}_X^{\heartsuit}$ is ordinary if and only if \mathfrak{X} is ordinary in the sense of Definition 1.5.1. This observation implies the following proposition for $n = 1$, by which one may consider the ordinariness of GL_n -do'pers may be thought of as a generalization of the classical ordinariness of algebraic curves in positive characteristic.

Proposition 2.4.1.

Let \mathcal{F}^{\heartsuit} be a GL_n -do'per on \mathfrak{X} . Then, \mathcal{F}^{\heartsuit} is ordinary if and only if \mathfrak{X} is ordinary and \mathcal{F}^{\heartsuit} is \otimes -ordinary. If, moreover, $n = 1$, then \mathcal{F}^{\heartsuit} is necessarily \otimes -ordinary.

Proof. Since we have verified the latter assertion, it suffices to prove the former assertion. Let us write $\mathcal{F}^{\heartsuit} := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n)$. The morphism $\mathcal{O}_X \oplus \mathcal{E}nd^{\otimes}(\mathcal{F}) \rightarrow \mathcal{E}nd(\mathcal{F})$ given by assigning $(a, v) \mapsto a \cdot \text{id}_{\mathcal{F}} + v$ is (since n is prime to p) an isomorphism that is compatible with $d \oplus \nabla_{\mathcal{F}}^{\text{ad}\otimes}$ and $\nabla_{\mathcal{F}}^{\text{ad}}$. This

isomorphism restricts to isomorphisms

$$(44) \quad \mathcal{O}_X \oplus \mathcal{E}nd^{\otimes}(\mathcal{F})^0 \xrightarrow{\sim} \mathcal{E}nd(\mathcal{F})^0 \quad \text{and} \quad \{0\} \oplus \mathcal{E}nd^{\otimes}(\mathcal{F})^1 \xrightarrow{\sim} \mathcal{E}nd(\mathcal{F})^1$$

that are compatible with $\nabla_{\mathcal{O}_X}^{\text{ad},1} \oplus \nabla_{\mathcal{F}}^{\text{ad},1}$ and $\nabla_{\mathcal{F}}^{\text{ad},1}$. Hence, we have a commutative square diagram

$$(45) \quad \begin{array}{ccc} f_*(\Omega_{X^{\log}/S^{\log}}) \oplus \text{Coker}(f_*(\nabla_{\mathcal{F}}^{\text{ad},1})) & \xrightarrow{\sim} & \text{Coker}(f_*(\nabla_{\mathcal{F}}^{\text{ad},1})) \\ \Theta_{\mathcal{O}_X^{\heartsuit}}^1 \oplus \Theta_{\mathcal{F}^{\heartsuit}}^{\otimes,1} \downarrow & & \downarrow \Theta_{\mathcal{F}^{\heartsuit}}^1 \\ f_*^{(1)}(\Omega_{X_S^{(1)\log}/S^{\log}}) \oplus f_*(\text{Coker}(\nabla_{\mathcal{F}}^{\text{ad},1})) & \xrightarrow{\sim} & f_*(\text{Coker}(\nabla_{\mathcal{F}}^{\text{ad},1})). \end{array}$$

This diagram shows that $\Theta_{\mathcal{F}^{\heartsuit}}^1$ is injective if and only if $\Theta_{\mathcal{O}_X^{\heartsuit}}^1$ is injective (equivalently, \mathfrak{X} is ordinary by the discussion preceding Proposition 2.4.1) and $\Theta_{\mathcal{F}^{\heartsuit}}^{\otimes,1}$ is injective. This completes the proof of Proposition 2.4.1. \square

2.5. According to Proposition 2.4.1, the study of the ordinariness of GL_n -do'pers reduces to the study of the \otimes -ordinariness. In this subsection, we shall explain that these conditions may be closely related to a local property of the moduli of GL_n -do'pers. Toward explaining it, we first recall (from [23]) several conditions each of which is equivalent to the \otimes -ordinariness (cf. Proposition 2.5.1 below).

For a GL_n -do'per $\mathcal{F}^{\heartsuit} := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n)$ on \mathfrak{X} , we shall denote by

$$(46) \quad \gamma_{\mathcal{F}^{\heartsuit}} : F_{X/S*}(\text{Ker}(\nabla_{\mathcal{F}})) \rightarrow F_{X/S*}(\mathcal{F}/\mathcal{F}^1)$$

the $\mathcal{O}_{X_S^{(1)}}$ -linear composite of the inclusion $F_{X/S*}(\text{Ker}(\nabla_{\mathcal{F}})) \hookrightarrow F_{X/S*}(\mathcal{F})$ and the quotient $F_{X/S*}(\mathcal{F}) \twoheadrightarrow F_{X/S*}(\mathcal{F}/\mathcal{F}^1)$.

Proposition 2.5.1.

Let \mathcal{F}^{\heartsuit} be as above, and suppose that S is affine. Then, the following three conditions (i)-(iii) are mutually equivalent to each other:

- (i) \mathcal{F}^{\heartsuit} is \otimes -ordinary.
- (ii) The morphism $\Theta_{\mathcal{F}^{\heartsuit}}^{\otimes,0}$ is injective.
- (iii) The dormant \mathfrak{sl}_n -oper associated with \mathcal{F}^{\heartsuit} has no nontrivial first order deformation. (Here, we refer to [23], Definition 2.2.1 (i) and Definition 3.6.1, for the definition of a dormant \mathfrak{g} -oper on a pointed stable curve for a semisimple Lie algebra \mathfrak{g} and to [23], § 4.3 and Corollary 4.13.3, for the discussion concerning the dormant \mathfrak{sl}_n -oper associated with a dormant GL_n -oper.)

Suppose further that $r = 0$, X is smooth over S , and is ordinary. Then, these conditions are also equivalent to the following condition:

(iv) *The equality*

$$(47) \quad \mathrm{Hom}_{\mathcal{O}_{X_S^{(1)}}}(F_{X/S*}(\mathrm{Ker}(\nabla_{\mathcal{F}})), \mathrm{Coker}(\gamma_{\mathcal{F}^\vee})) = 0$$

is satisfied.

Proof. The equivalence (i) \Leftrightarrow (ii) follows from [23], Lemma 5.2.1 (i). The equivalence (ii) \Leftrightarrow (iii) follows from [23], Theorem D and Corollary 5.12.1 (i). Moreover, the equivalence (iii) \Leftrightarrow (iv) follows from [23] Proposition 8.3.3 and Proposition 8.4.1 (and the discussion in [23], § 8.1). \square

By the equivalence (i) \Leftrightarrow (iii) in Proposition 2.5.1, the \circledast -ordinariness of GL_n -do'pers may be linked to the study in the paper [23] concerning the geometric structure of the moduli stack of dormant \mathfrak{sl}_n -opers. In fact, denote by

$$(48) \quad \mathfrak{Op}_{\mathrm{GL}_n, g, r}^{\mathrm{Zzz}\dots}$$

be the sheaf associated with the étale presheaf on $\overline{\mathfrak{M}}_{g, r}$ which, to any k -scheme T equipped with an étale morphism $T \rightarrow \overline{\mathfrak{M}}_{g, r}$ classifying pointed stable curve \mathfrak{Y} over T , assigns the set of equivalence classes of GL_n -do'pers on \mathfrak{Y} . Also, denote by

$$(49) \quad \mathfrak{Op}_{\mathfrak{sl}_n, g, r}^{\mathrm{Zzz}\dots}$$

the étale sheaf on $\overline{\mathfrak{M}}_{g, r}$ which, to any T (as well as \mathfrak{Y}) as above, assigns the set of isomorphism classes of dormant \mathfrak{sl}_n -opers on \mathfrak{Y} . By [23], Theorem C, Theorem D, and the isomorphism (11), these sheaves may be represented by finite (relative) $\overline{\mathfrak{M}}_{g, r}$ -schemes and the assignment from each GL_n -do'per to its associated dormant \mathfrak{sl}_n -oper defines an isomorphism

$$(50) \quad \Lambda : \mathfrak{Op}_{\mathrm{GL}_n, g, r}^{\mathrm{Zzz}\dots} \xrightarrow{\sim} \mathfrak{Op}_{\mathfrak{sl}_n, g, r}^{\mathrm{Zzz}\dots}$$

of (relative) $\overline{\mathfrak{M}}_{g, r}$ -schemes. Let \mathcal{F}^\vee be a GL_n -do'per \mathcal{F}^\vee on \mathfrak{X} and $s : S \rightarrow \mathfrak{Op}_{\mathrm{GL}_n, g, r}^{\mathrm{Zzz}\dots}$ the S -rational point of $\mathfrak{Op}_{\mathrm{GL}_n, g, r}^{\mathrm{Zzz}\dots}$ classifying \mathcal{F}^\vee . Then, the equivalence (i) \Leftrightarrow (iii) in Proposition 2.5.1 implies that \mathcal{F}^\vee is \circledast -ordinary if and only if the image of the composite $\Lambda \circ s$ lies in the unramified locus of $\mathfrak{Op}_{\mathfrak{sl}_n, g, r}^{\mathrm{Zzz}\dots}$ relative to $\overline{\mathfrak{M}}_{g, r}$.

By applying results in [23], we have the following proposition.

Proposition 2.5.2.

(i) *Denote by*

$$(51) \quad \circledast \mathfrak{Op}_{\mathrm{GL}_n, g, r}^{\mathrm{Zzz}\dots}$$

the locus of $\mathfrak{Op}_{\mathrm{GL}_n, g, r}^{\mathrm{Zzz}\dots}$ classifying \circledast -ordinary GL_n -do'pers. Then, it forms an open substack which is dense in any irreducible component of $\mathfrak{Op}_{\mathrm{GL}_n, g, r}^{\mathrm{Zzz}\dots}$ that dominates $\overline{\mathfrak{M}}_{g, r}$, and $\circledast \mathfrak{Op}_{\mathrm{GL}_n, g, r}^{\mathrm{Zzz}\dots} / \overline{\mathfrak{M}}_{g, r}$ is finite and

étale. In particular, any GL_n -do'per on the pointed proper smooth curve classified by the generic point of $\mathfrak{M}_{g,r}$ ($\subseteq \overline{\mathfrak{M}}_{g,r}$) is ordinary (and hence, \otimes -ordinary).

- (ii) *Any GL_n -do'per on a totally degenerate pointed stable curve (e.g., the projective line together with 3 marked points) is necessarily ordinary, and hence, \otimes -ordinary (cf. [23], Definition 6.3.1, for the definition of a totally degenerate pointed stable curve).*

Proof. Assertions (i) and (ii) follows from [23], Theorem F, and the fact that any pointed totally degenerate stable curve is ordinary in the sense of Definition 1.5.1. \square

Corollary 2.5.3.

Let S^{\log} be a log scheme over k such that $S = \mathrm{Spec}(R)$ for some local ring R over k . Also, let \mathfrak{X} be a pointed stable curve over S^{\log} whose spacial fiber is totally degenerate. Then, any GL_n -do'per on \mathfrak{X} is ordinary, and hence, \otimes -ordinary.

Proof. The assertion follows directly from Proposition 2.5.2 (i) and (ii). \square

2.6. We shall analyze the behavior, according to the clutching morphism of the underlying pointed stable curves, of the \otimes -ordinariness.

Let S^{\log} be a log scheme over k and $\mathfrak{X} := (X/S, \{\sigma_i\}_{i=1}^r)$ a pointed stable curve over S^{\log} of type (g, r) (where $2g - 2 + r > 0$). Suppose that \mathfrak{X} may be obtained by gluing together m (≥ 1) pointed stable curves $\mathfrak{X}_j := (X_j/S, \{\sigma_{j,i}\}_{i=1}^{r_j})$ ($j = 1, \dots, m$) along some marked points in $\bigcup_{j=1}^m \{\sigma_{j,i}\}_{i=1}^{r_j}$. (More precisely, \mathfrak{X} is obtained by gluing together $\{\mathfrak{X}_j\}_{j=1}^m$ by means of a certain *clutching data* for a pointed stable curve of type (g, r) . We refer to [23], Definition 6.1.1, for the definition of a clutching data.) In particular, for each j , we have a natural morphism $\mathfrak{Clut}_j : X_j \rightarrow X$ of S -schemes. Let us consider the log structure on X_j pulled-back from the log structure of $X^{\mathfrak{X}\text{-log}}$ via \mathfrak{Clut}_j . We shall denote the resulting log scheme by $X_j^{\mathfrak{X}\text{-log}}$. The structure morphism $X_j \rightarrow S$ of X_j/S extends to a morphism $X_j^{\mathfrak{X}\text{-log}} \rightarrow S^{\mathfrak{X}\text{-log}}$ of log schemes. Moreover, the natural morphism $X_j^{\mathfrak{X}\text{-log}} \rightarrow X_j$ of log schemes (where we consider X_j as being equipped with the trivial log structure) extends to a commutative square diagram

$$(52) \quad \begin{array}{ccc} X_j^{\mathfrak{X}\text{-log}} & \xrightarrow{w_j^{\log}} & X_j^{\mathfrak{X}_j\text{-log}} \\ \downarrow & & \downarrow \\ S^{\mathfrak{X}\text{-log}} & \xrightarrow{v_j^{\log}} & S^{\mathfrak{X}_j\text{-log}}, \end{array}$$

where the underlying morphisms w_j and v_j of the horizontal arrows coincide with the identity morphisms of X_j and S respectively. One verifies that the morphism

$$(53) \quad w_{v_j} : X_j^{\mathfrak{X}\text{-log}} \rightarrow X_j^{\mathfrak{X}_j\text{-log}} \times_{S^{\mathfrak{X}_j\text{-log}}} S^{\mathfrak{X}\text{-log}}$$

over $S^{\mathfrak{X}\text{-log}}$ induced by the diagram (52) is log étale.

Now, suppose that we are given a GL_n -do'per \mathcal{F}^\heartsuit on \mathfrak{X} . By restricting via $\mathfrak{C}\mathfrak{h}\mathfrak{u}\mathfrak{t}_j$, we obtain, from \mathcal{F}^\heartsuit , a GL_n -do'per $\mathcal{F}|_{X_j^{\mathfrak{X}\text{-log}}}^\heartsuit$ on $X_j^{\mathfrak{X}\text{-log}}/S^{\mathfrak{X}\text{-log}}$. Moreover, since w_j^{log} induces a canonical isomorphism $\Omega_{X^{\mathfrak{X}\text{-log}}/S^{\mathfrak{X}\text{-log}}} \xrightarrow{\sim} \Omega_{X_j^{\mathfrak{X}\text{-log}}/S^{\mathfrak{X}_j\text{-log}}}$, $\mathcal{F}|_{X_j^{\mathfrak{X}\text{-log}}}^\heartsuit$ may be naturally identified with a GL_n -do'per

$$(54) \quad \mathcal{F}|_{\mathfrak{X}_j}^\heartsuit$$

on \mathfrak{X}_j/S . The following proposition (which will be used in the proof of Theorem B) is verified.

Proposition 2.6.1.

Let \mathfrak{X} , \mathfrak{X}_j ($j = 1, \dots, m$), and \mathcal{F}^\heartsuit be as above. Then, \mathcal{F}^\heartsuit is \otimes -ordinary if and only if for any $j \in \{1, \dots, m\}$ the GL_n -do'per $\mathcal{F}|_{\mathfrak{X}_j}^\heartsuit$ is \otimes -ordinary.

Proof. The assertion follows from the equivalence (i) \Leftrightarrow (iii) in Proposition 2.5.1 and [23], Theorem 6.2.2. \square

2.7. The proof of Theorem A. Now, we shall prove one of our main results, Theorem A. Let \mathfrak{X} , (\mathfrak{Y}, w) , and \mathcal{F}^\heartsuit be as in §0.2. It follows from the assumption and Proposition 2.4.1 that \mathfrak{Y} is ordinary and $w^*(\mathcal{F})^\heartsuit$ is \otimes -ordinary. For completing the proof, it suffices to prove that \mathfrak{X} is ordinary and \mathcal{F}^\heartsuit is \otimes -ordinary (by Proposition 2.4.1 again).

First, we shall show that \mathfrak{X} is ordinary. Consider the square diagram

$$(55) \quad \begin{array}{ccc} \Gamma(X, \Omega_{X^{\mathrm{log}}/k}) & \xrightarrow{\Gamma(X_k^{(1)}, C_{X/k})} & \Gamma(X_k^{(1)}, \Omega_{X_k^{(1)\mathrm{log}}/k}) \\ \downarrow & & \downarrow \\ \Gamma(Y, \Omega_{Y^{\mathrm{log}}/k}) & \xrightarrow{\Gamma(Y_k^{(1)}, C_{Y/k})} & \Gamma(Y_k^{(1)}, \Omega_{Y_k^{(1)\mathrm{log}}/k}), \end{array}$$

where the left-hand and right-hand vertical arrows are the morphisms induced from w and $w^{(1)}$ respectively. By Proposition 1.5.2, the ordinariness of \mathfrak{Y} implies the injectivity of the lower horizontal arrow in (55). Hence, since the two vertical arrows in (55) are injective, the upper horizontal arrow is injective. By Proposition 1.5.2 again, \mathfrak{X} turns out to be ordinary.

Next, we shall show that \mathcal{F}^\heartsuit is \otimes -ordinary. Recall that the isomorphism

$$(56) \quad w^{(1)*}(F_{X/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}}))) \xrightarrow{\sim} F_{Y/k*}(\mathrm{Ker}(\nabla_{w^*(\mathcal{F})}))$$

obtained by Proposition 1.6.2. Also, the functorial isomorphism (27) induces an isomorphism

$$(57) \quad w^{(1)*}(F_{X/k*}(\mathcal{F}/\mathcal{F}^1)) \xrightarrow{\sim} F_{Y/k*}(w^*(\mathcal{F})/w^*(\mathcal{F})^1).$$

The above two isomorphisms make the square diagram

$$(58) \quad \begin{array}{ccc} w^{(1)*}(F_{X/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}}))) & \xrightarrow{w^{(1)*}(\gamma_{\mathcal{F}^\heartsuit})} & w^{(1)*}(F_{X/k*}(\mathcal{F}/\mathcal{F}^1)) \\ (56) \downarrow \wr & & \wr \downarrow (57) \\ F_{Y/k*}(\mathrm{Ker}(\nabla_{w^*(\mathcal{F})})) & \xrightarrow{\gamma_{w^*(\mathcal{F})}^\heartsuit} & F_{Y/k*}(w^*(\mathcal{F})/w^*(\mathcal{F})^1) \end{array}$$

commute. Hence, we have an isomorphism

$$(59) \quad w^{(1)*}(\mathrm{Coker}(\gamma_{\mathcal{F}^\heartsuit})) \xrightarrow{\sim} \mathrm{Coker}(\gamma_{w^*(\mathcal{F})}^\heartsuit).$$

By means of (56) and (59), we obtain a composite injection

$$(60) \quad \begin{aligned} & \mathrm{Hom}_{\mathcal{O}_{X(1)}}(F_{X/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}})), \mathrm{Coker}(\gamma_{\mathcal{F}^\heartsuit})) \\ & \hookrightarrow \mathrm{Hom}_{\mathcal{O}_X}(w^{(1)*}(F_{X/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}}))), w^{(1)*}(\mathrm{Coker}(\gamma_{\mathcal{F}^\heartsuit}))) \\ & \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(F_{Y/k*}(\mathrm{Ker}(\nabla_{w^*(\mathcal{F})})), \mathrm{Coker}(\gamma_{w^*(\mathcal{F})}^\heartsuit)). \end{aligned}$$

But, by the equivalence (i) \Leftrightarrow (iv) in Proposition 2.5.1, the hypothesis that $w^*(\mathcal{F})^\heartsuit$ is \otimes -ordinary implies the equality

$$(61) \quad \mathrm{Hom}_{\mathcal{O}_X}(F_{Y/k*}(\mathrm{Ker}(\nabla_{w^*(\mathcal{F})})), \mathrm{Coker}(\gamma_{w^*(\mathcal{F})}^\heartsuit)) = 0.$$

Consequently, the composite injection (60) shows the equality

$$(62) \quad \mathrm{Hom}_{\mathcal{O}_{X(1)}}(F_{X/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}})), \mathrm{Coker}(\gamma_{\mathcal{F}^\heartsuit})) = 0,$$

equivalently, \mathcal{F}^\heartsuit is \otimes -ordinary by Proposition 2.5.1 again. This completes the proof of Theorem A.

3. CYCLIC LOG ÉTALE COVERINGS OF A POINTED PROJECTIVE LINE

In this section, we consider GL_n -do'pers on the projective line with some marked points. The goal of this section is to prove (cf. Proposition 3.3.1) that the pull-back of any GL_n -do'per on the projective line with three marked points via a cyclic log étale covering is necessarily \otimes -ordinary. This assertion will be the key ingredient of the proof of Theorem B.

3.1. For each value $s \in k \sqcup \{\infty\}$, denote by $[s]$ the k -rational point of the projective line

$$(63) \quad \mathbb{P} := \text{Proj}(k[x, y])$$

over k determined by s . Let l be a positive integer prime to p , and denote by $\mu_l := \{\zeta_1, \dots, \zeta_l\}$ the group of l -power roots of unity. The collection of data

$$(64) \quad \mathfrak{P}^{(l+2)\text{pt}} := (\mathbb{P}/k, \{[0], [\zeta_1], \dots, [\zeta_l], [\infty]\})$$

forms a pointed stable curve of type $(0, l+2)$. In particular,

$$(65) \quad \mathfrak{P}^{3\text{pt}} := (\mathbb{P}/k, \{[0], [1], [\infty]\})$$

(i.e, the $\mathfrak{P}^{(l+2)\text{pt}}$ of the case $l = 1$) is a *unique* (up to isomorphism) pointed stable curve of type $(0, 3)$.

3.2. The Frobenius twist $\mathbb{P}_k^{(1)}$ of \mathbb{P}/k is evidently isomorphic to \mathbb{P} . The discussion in this subsection will be applied to the discussion in §3.3, where we deal with $\mathbb{P}_k^{(1)}$, via a natural identification $\mathbb{P}_k^{(1)} \cong \mathbb{P}$.

Let n be a positive integer. Recall the Birkhoff-Grothendieck's theorem, which asserts that for any vector bundle \mathcal{V} on \mathbb{P} of rank n is isomorphic to a direct sum of n line bundles:

$$(66) \quad \mathcal{V} \cong \bigoplus_{j=1}^n \mathcal{O}_{\mathbb{P}}(w_j),$$

where $w_{j_1} \leq w_{j_2}$ if $j_1 < j_2$. The nondecreasing sequence of integers $(w_j)_{j=1}^n$ depends only on the isomorphism class of the vector bundle \mathcal{V} .

Definition 3.2.1.

In the above situation, we shall say that \mathcal{V} is **of type** $(w_j)_{j=1}^n$. Also, we shall say that \mathcal{V} is **of homogeneous type** if it is of type $(w_j)_{j=1}^n$ with $w_1 = w_n$ (equivalently, $w_1 = w_2 = \dots = w_n$).

By induction on n , one may verify immediately the following lemma (cf. [23]), Lemma 7.7.1.

Lemma 3.2.2.

Suppose that we are given, for $\square = 1, 2$, a rank n vector bundle \mathcal{V}_{\square} on \mathbb{P} of type $(w_{\square,j})_{j=1}^n$, where $\{w_{\square,j}\}_{1 \leq j \leq n}$ denotes a nondecreasing sequence of integers, and given an injection $\mathcal{V}_1 \hookrightarrow \mathcal{V}_2$ of $\mathcal{O}_{\mathbb{P}}$ -modules. Then, for any $j \in \{1, \dots, n\}$, the inequality $w_{1,j} \leq w_{2,j}$ is satisfied.

The above lemma deduces immediately the following lemma (from an argument similar to the argument in the proof of [23], Lemma 7.7.2), which will be used in the proof of Proposition 3.3.1.

Lemma 3.2.3.

Let s be an integer and $\{\mathcal{V}_l\}_{l \in \mathbb{Z}_{\geq 0}}$ a set of rank n vector bundles (indexed by the set of nonnegative integers $\mathbb{Z}_{\geq 0}$) on \mathbb{P} satisfying the following two conditions:

- each \mathcal{V}_l is of degree $s + l$ and of type $(w_{l,j})_{j=1}^n$, where $\{w_{l,j}\}_{j=1}^n$ denotes a nondecreasing sequence of integers (hence $\sum_{j=1}^n w_{l,j} = s + l$);
- the inequality $w_{l,n} - w_{l,1} \leq 1$ (for each $l \geq 0$) is satisfied.

Also, suppose that we are given a sequence of $\mathcal{O}_{\mathbb{P}}$ -linear injections

$$(67) \quad \mathcal{V}_0 \hookrightarrow \mathcal{V}_1 \hookrightarrow \mathcal{V}_2 \hookrightarrow \mathcal{V}_3 \hookrightarrow \cdots$$

Then, there exists $l_0 \in \mathbb{Z}_{\geq 0}$ such that \mathcal{V}_{l_0} is of homogeneous type.

Proposition 3.2.4.

Let $\mathcal{F}^\heartsuit := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n)$ be a GL_n -doper on $\mathfrak{P}^{(l+2)\mathrm{pt}}$ and $\{w_l\}_{l=1}^n$ a nondecreasing sequence of integers. Suppose that the rank n vector bundle $F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}}))$ on $\mathbb{P}_k^{(1)}$ is of type $(w_l)_{l=1}^n$, i.e., decomposes into a direct sum of n line bundles:

$$(68) \quad F_{\mathbb{P}^1/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}})) \xrightarrow{\sim} \bigoplus_{l=1}^n \mathcal{O}_{\mathbb{P}_k^{(1)}}(w_l).$$

Then, we have the inequality $w_n - w_1 \leq l + 1$, equivalently, $|w_i - w_j| \leq l + 1$ for any $i, j \in \{1, \dots, n\}$.

Proof. According to the decomposition (68), the rank n vector bundle

$$(69) \quad \mathcal{A}_{\mathrm{Ker}(\nabla_{\mathcal{F}})} := F_{\mathbb{P}/k}^*(F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}})))$$

on \mathbb{P} decomposes into the direct sum

$$(70) \quad \mathcal{A}_{\mathrm{Ker}(\nabla_{\mathcal{F}})} \xrightarrow{\sim} \bigoplus_{l=1}^n \mathcal{O}_{\mathbb{P}}(p \cdot w_l).$$

Recall from [23], § 7.2, (777), that $\mathcal{A}_{\mathrm{Ker}(\nabla_{\mathcal{F}})}$ admits a canonical decreasing filtration $\{\mathcal{A}_{\mathrm{Ker}(\nabla_{\mathcal{F}})}^j\}_{j=0}^n$ with $\mathrm{gr}_{\mathcal{F}}^j := \mathcal{A}_{\mathrm{Ker}(\nabla_{\mathcal{F}})}^j / \mathcal{A}_{\mathrm{Ker}(\nabla_{\mathcal{F}})}^{j+1}$ ($j = 0, \dots, n-1$). It follows from [23], Proposition 7.3.3 and the latter assertion of [23], Corollary 7.3.4 of the case where $(g, r) = (0, 3)$ (cf. [23], the proof of Proposition 7.6.3) that

$$(71) \quad \deg(\mathrm{gr}_{\mathcal{F}}^{n-1}) < \deg(\mathrm{gr}_{\mathcal{F}}^{n-2}) < \cdots < \deg(\mathrm{gr}_{\mathcal{F}}^0).$$

Now, let us write

$$(72) \quad \begin{aligned} \xi' : \mathcal{O}_{\mathbb{P}}(p \cdot w_n) &\hookrightarrow \bigoplus_{l=1}^n \mathcal{O}_{\mathbb{P}}(p \cdot w_l) \xrightarrow{(70)^{-1}} \mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})} \\ (\text{resp., } \xi'' : \mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})} &\xrightarrow{(70)} \bigoplus_{l=1}^n \mathcal{O}_{\mathbb{P}}(p \cdot w_l) \twoheadrightarrow \mathcal{O}_{\mathbb{P}}(p \cdot w_1)), \end{aligned}$$

where the first (resp., second) arrow denotes the inclusion into the n -th factor (resp., the projection onto the 1-st factor). Also, write

$$(73) \quad \begin{aligned} n' &:= \max\{j \mid \text{Im}(\xi') \subseteq \mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})}^j\} \\ (\text{resp., } n'' &:= \max\{j \mid \xi'(\mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})}^j) \neq 0\}). \end{aligned}$$

Then, ξ' (resp., ξ'') induces a *nonzero* morphism

$$(74) \quad \begin{aligned} \bar{\xi}' : \mathcal{O}_{\mathbb{P}}(p \cdot w_n) &\rightarrow \mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})}^{n'} / \mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})}^{n'+1} (= \text{gr}_{\mathcal{F}}^{n'}) \\ (\text{resp., } \bar{\xi}'' : (\text{gr}_{\mathcal{F}}^{n''} =) &\mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})}^{n''} / \mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})}^{n''+1} \rightarrow \mathcal{O}_{\mathbb{P}}(p \cdot w_1)) \end{aligned}$$

between line bundles (on the smooth curve \mathbb{P}^1). In particular $\bar{\xi}'$ and $\bar{\xi}''$ are injective, and hence, we have

$$(75) \quad p \cdot w_n = \deg(\mathcal{O}_{\mathbb{P}}(p \cdot w_n)) \leq \deg(\text{gr}_{\mathcal{F}}^{n'}) \leq \deg(\text{gr}_{\mathcal{F}}^0)$$

and

$$(76) \quad p \cdot w_1 = \deg(\mathcal{O}_{\mathbb{P}}(p \cdot w_1)) \geq \deg(\text{gr}_{\mathcal{F}}^{n''}) \geq \deg(\text{gr}_{\mathcal{F}}^{n-1}),$$

where the last inequalities in both (75) and (76) follow from (71). But, it follows from the latter assertion of [23], Corollary 7.3.4, that

$$(77) \quad \begin{aligned} &\deg(\text{gr}_{\mathcal{F}}^0) - \deg(\text{gr}_{\mathcal{F}}^{n-1}) \\ &= -(n-1) \cdot (2 \cdot 0 - 2 + l + 2) + \sum_{i=1}^{l+2} (w_{i,n} - w_{i,1}) \\ &< -(n-1) \cdot l + (l+2) \cdot p. \end{aligned}$$

By combining (75), (76), and (77), we obtain

$$(78) \quad w_n - w_1 \leq \frac{1}{p} \cdot (\deg(\text{gr}_{\mathcal{F}}^0) - \deg(\text{gr}_{\mathcal{F}}^{n-1})) < l + 2 - \frac{(n-1) \cdot l}{p}.$$

This implies (since $w_n - w_1 \in \mathbb{Z}$) that $w_n - w_1 \leq l + 1$, as desired. \square

3.3. Consider the endomorphism $\pi : \mathbb{P} \rightarrow \mathbb{P}$ of \mathbb{P} corresponding to the k -algebra endomorphism of $k[x, y]$ given by assigning $x \mapsto x^l$ and $y \mapsto y^l$. This morphism is unramified over $\mathbb{P} \setminus \{[0], [\infty]\}$ and satisfies that $\pi^{-1}([0]) = \{[0]\}$, $\pi^{-1}([\infty]) = \{[\infty]\}$, and $\pi^{-1}([1]) = \{[\zeta_1], \dots, [\zeta_l]\}$. Also, it extends to a log étale Galois covering

$$(79) \quad \pi^{\log} : \mathbb{P}^{\mathfrak{P}^{(l+2)\text{pt}}-\log} \rightarrow \mathbb{P}^{\mathfrak{P}^{3\text{pt}}-\log}$$

over k with Galois group μ_l .

Proposition 3.3.1.

Let $\mathcal{F}^\heartsuit := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n)$ be a GL_n -do'per on $\mathfrak{P}^{3\text{pt}}$ (which is \otimes -ordinary due to Proposition 2.5.1 (i) or (ii)). Then, the pull-back $\pi^*(\mathcal{F})^\heartsuit$ via π^{\log} (i.e., the pull-back of \mathcal{F}^\heartsuit via $(\pi^{\log}, \text{id}_k)$ in the sense of Definition 2.1.3) is a \otimes -ordinary GL_n -do'per on $\mathfrak{P}^{(l+2)\text{pt}}$.

Proof. By Remark 2.3.1.1 and Lemma 3.3.3 proved below, one may assume, after possibly twisting by some integrable line bundle with vanishing p -curvature, that $F_{\mathbb{P}/k*}(\text{Ker}(\nabla_{\mathcal{F}}))$ is of homogenous type. Since $\pi^{(1)*}(F_{\mathbb{P}/k*}(\text{Ker}(\nabla_{\mathcal{F}}))) \cong F_{\mathbb{P}/k*}(\text{Ker}(\nabla_{\pi^*(\mathcal{F})}))$ by Lemma 3.3.2 proved below, $F_{\mathbb{P}/k*}(\text{Ker}(\nabla_{\pi^*(\mathcal{F})}))$ is of homogenous type, and hence, $\mathcal{E}nd(F_{\mathbb{P}/k*}(\text{Ker}(\nabla_{\pi^*(\mathcal{F})})))$ is isomorphic to a direct sum of finite copies of $\mathcal{O}_{\mathbb{P}_k^{(1)}}$. This implies the equality

$$(80) \quad H^1(\mathbb{P}_k^{(1)}, \mathcal{E}nd(F_{\mathbb{P}/k*}(\text{Ker}(\nabla_{\pi^*(\mathcal{F})})))) = 0.$$

Here, recall from well-known generalities of deformation theory that the space of first order deformations of $F_{\mathbb{P}/k*}(\text{Ker}(\nabla_{\pi^*(\mathcal{F})}))$ is canonically isomorphic to $H^1(\mathbb{P}_k^{(1)}, \mathcal{E}nd(F_{\mathbb{P}/k*}(\text{Ker}(\nabla_{\pi^*(\mathcal{F})}))))$. Thus, $F_{\mathbb{P}/k*}(\text{Ker}(\nabla_{\pi^*(\mathcal{F})}))$ has no nontrivial first order deformation. The assertion follows from this fact together with [23], Corollary 7.5.2. \square

To complete the proof of Proposition 3.3.1, it remains to prove Lemmas 3.3.2 and 3.3.3. We first prove Lemma 3.3.2, which asserts that there exists an isomorphism of the form (28) where w is replaced with our covering π^{\log} (which is not étale but log étale).

Lemma 3.3.2.

Let $(\mathcal{V}, \nabla_{\mathcal{V}})$ be an integrable vector bundle on $\mathfrak{P}^{3\text{pt}}$ with vanishing p -curvature. Then, the natural morphism

$$(81) \quad \pi^{(1)*}(F_{\mathbb{P}/k*}(\text{Ker}(\nabla_{\mathcal{V}}))) \rightarrow F_{\mathbb{P}/k*}(\text{Ker}(\nabla_{\pi^*(\mathcal{V})}))$$

of $\mathcal{O}_{\mathbb{P}}$ -modules is an isomorphism.

Proof. Since (81) is an isomorphism on the open subset $\mathbb{P}^{(1)} \setminus \{[0], [\infty]\}$ (cf. (28)), it suffices to prove that (81) is an isomorphism on the formal neighborhoods of $[0]$ and $[\infty]$.

By similarity of the argument, we shall only consider the case of $[0]$. Let us fix a local parameter t of \mathbb{P} at $[0]$. Then, the formal neighborhood of $[0]$ in \mathbb{P} (resp., $\mathbb{P}_k^{(1)}$) is isomorphic to $\mathrm{Spf}(k[[t]])$ (resp., $\mathrm{Spf}(k[[t^p]])$) and the local parameter t gives an identification $\Omega_{\mathbb{P}/k}|_{\mathrm{Spf}(k[[t]])} \xrightarrow{\sim} \widetilde{k[[t]]}$ (where $\widetilde{(-)}$ denotes the coherent sheaf associated with $(-)$). It follows from [18], Corollary 2.10 (or, [23], the discussion following Lemma 7.3.2) that the t^p -adic completion of $(F_{\mathbb{P}/k*}(\mathcal{V}), F_{\mathbb{P}/k*}(\nabla_{\mathcal{V}}))$ is isomorphic to the direct sum $\widetilde{k[[t]]}^{\oplus n}$ together with a $\widetilde{k[[t^p]]}$ -linear morphism $\bigoplus_{j=1}^n \nabla_{m_j}$, where, for each $m \in \mathbb{Z}$, ∇_m denotes the map $\widetilde{k[[t]]} \rightarrow \widetilde{k[[t]]}$ determined by $a \mapsto t \cdot \frac{\partial a}{\partial t} - m \cdot a$ for any $a \in k[[t]]$. Hence, to complete the proof, one may assume, without loss of generality, that $(\mathcal{V}, \nabla_{\mathcal{V}})|_{\mathrm{Spf}(k[[t]])} \cong (\widetilde{k[[t]]}, \nabla_m)$ (where $m \in \mathbb{Z}$). We shall identify, in a natural fashion, the formal neighborhood of $[0]$ in the domain \mathbb{P} of π (resp., the domain $\mathbb{P}^{(1)}$ of $\pi^{(1)}$) with $\mathrm{Spf}(k[[t^{\frac{1}{l}}]])$ (resp., $\mathrm{Spf}(k[[t^{\frac{p}{l}}]])$). The ring homomorphism $k[[t]] \rightarrow k[[t^{\frac{1}{l}}]]$ (resp., $k[[t^p]] \rightarrow k[[t^{\frac{p}{l}}]])$ obtained as the formal completion of π (resp., $\pi^{(1)}$) at $[0]$ may be given by assigning $t \mapsto (t^{\frac{1}{l}})^l$ (resp., $t^p \mapsto (t^{\frac{p}{l}})^l$). Since $F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}}))|_{\mathrm{Spf}(k[[t^p]])}$ is isomorphic to $\widetilde{k[[t^p]]} \cdot t^m$, we have

$$(82) \quad \pi^{(1)*}(F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}})))|_{\mathrm{Spf}(k[[t^{\frac{p}{l}}]])} \xrightarrow{\sim} \widetilde{k[[t^{\frac{p}{l}}]]} \cdot t^m.$$

On the other hand, one verifies that $\nabla_{\pi^*(\mathcal{V})}|_{\mathrm{Spf}(k[[t^{\frac{1}{l}}]])}$ is isomorphic to the map given by assigning $a \mapsto t^{\frac{1}{l}} \cdot \frac{\partial a}{\partial(t^{\frac{1}{l}})} - lm \cdot a$ ($a \in k[[t^{\frac{1}{l}}]]$). Hence, we have

$$(83) \quad F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\pi^*(\mathcal{V})}))|_{\mathrm{Spf}(k[[t^{\frac{p}{l}}]])} \cong \widetilde{k[[t^{\frac{p}{l}}]]} \cdot (t^{\frac{1}{l}})^{lm}.$$

By (82) and (83), the morphism (81) is an isomorphism on the formal neighborhood of $[0]$. This completes the proof of Lemma 3.3.2. \square

By means of Lemma 3.3.2, we prove the following Lemma 3.3.3.

Lemma 3.3.3.

For any GL_n -do'per $\mathcal{F}^{\heartsuit} := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}^j\}_{j=0}^n)$ on $\mathfrak{P}^{3\mathrm{pt}}$, there exists a GL_n -do'per $\mathcal{G}^{\heartsuit} := (\mathcal{G}, \nabla_{\mathcal{G}}, \{\mathcal{G}^j\}_{j=0}^n)$ on $\mathfrak{P}^{3\mathrm{pt}}$ equivalent to \mathcal{F}^{\heartsuit} such that the vector bundle $F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\mathcal{G}}))$ on $\mathbb{P}_k^{(1)}$ is of homogeneous type.

Proof. We shall apply the discussion in the proof of [23], Proposition 7.7.4, to \mathcal{F}^{\heartsuit} . Here, note that there necessarily exists a $(n, 1)$ -determinant data \mathbb{U} for X^{\log} over S^{\log} (cf. [23], Definition 4.9, for the definition of a $(n, 1)$ -determinant data) for which \mathcal{F}^{\heartsuit} is isomorphic to some dormant $(\mathrm{GL}_n, 1, \mathbb{U})$ -oper. By applying the discussion in *loc. cit.*, we obtain a collection of data

$$(84) \quad \{\mathcal{F}_m^{\heartsuit}\}_{m \in \mathbb{Z}_{\geq 0}}, \quad \{i_m\}_{m \in \mathbb{Z}_{\geq 0}},$$

consisting of

- GL_n -do'pers $\mathcal{F}_m^\heartsuit := (\mathcal{F}_m, \nabla_{\mathcal{F}_m}, \{\mathcal{F}_m^j\}_{j=0}^n)$ ($m \in \mathbb{Z}_{\geq 0}$) on $\mathfrak{P}^{\mathrm{3pt}}$ which are equivalent to \mathcal{F}^\heartsuit and satisfy the equalities

$$(85) \quad \deg(F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}_{m+1}}))) = \deg(F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}_m}))) + 1,$$

and

- a sequence of $\mathcal{O}_{\mathbb{P}_k^{(1)}}$ -linear injections

$$(86) \quad F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}_0})) \hookrightarrow F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}_1})) \hookrightarrow F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}_3})) \hookrightarrow \cdots$$

Suppose that there exists $m_1 \geq 0$ for which $F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}_{m_1}}))$ is of type $(w_{m_1,j})_{j=1}^n$ with $w_{m_1,n} - w_{m_1,1} \geq 2$. Let us take a positive integer l with $l \geq 2$ and take π^{log} as at the beginning of § 3.3 for this l . Then, $F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\pi^*(\mathcal{F}_{m_1})}))$ (which is isomorphic to $\pi^{(1)*}(F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}_{m_1}})))$ by Lemma 3.3.2) is of type $(l \cdot w_{m_1,j})_{j=1}^n$. But, it follows from Proposition 3.2.4 that

$$(87) \quad (l \cdot 2 \leq) l \cdot w_{m_1,n} - l \cdot w_{m_1,1} \leq l + 1.$$

This is a contradiction, and implies that $F_{\mathbb{P}/k*}(\mathrm{Ker}(\nabla_{\mathcal{F}_m}))$ (for any m) is of type $(w_j)_{j=1}^n$ with $w_n - w_1 \leq 1$. Thus, by Lemma 3.2.3, there exists $m_2 \geq 0$ such that the GL_n -do'per $\mathcal{F}_{m_2}^\heartsuit$ satisfies the required conditions, as desired. \square

4. GL_n -DO'PERS ON A TOTALLY DEGENERATE CURVE OF GENUS 1

The goal of this section is to prove Theorem B. To this end, we apply a usual degeneration technique for underlying curves of GL_n -do'pers. By means of the result of the previous section, we first prove Theorem B for the case of elliptic curves and abelian coverings given by multiplication by integers (cf. Proposition 4.2.2). Then, we complete the proof of Theorem B by applying this tentative result and considering a stable curve obtained by gluing together some elliptic curves.

4.1. In this subsection, we consider a certain stable curve obtained by gluing together some copies of the projective line, and consider the log structure of the projective line restricted from the natural log structure of this stable curve.

For a positive integer m , we shall denote by

$$(88) \quad \mathfrak{C}_m^l := (C_m/k, \{\sigma_{m,i}^l\}_{i=1}^{ml})$$

the pointed stable curve over k of type $(1, ml)$ determined uniquely by the following two conditions:

(i) C_m is a (proper) semistable curve over k of the form

$$(89) \quad C_m := \bigcup_{j=1}^m P_j,$$

where each P_j is (a copy of) the projective line \mathbb{P} with $\{\sigma_{m,i}^l\}_{(j-1)l < i \leq jl} \subseteq P_j(k)$;

(ii) there exist mutually distinct k -rational points of C_m :

$$(90) \quad \delta_1 \cdots, \delta_m$$

such that

$$(91) \quad P_i \cap P_j = \begin{cases} \delta_i & \text{if } j = i + 1 \bmod m \\ \emptyset & \text{if otherwise.} \end{cases}$$

(Hence, the set $\{\delta_1 \cdots, \delta_m\}$ coincides with the set of nodes of C_m).

In particular, for each $j = 1, \dots, m$, we obtain (after ordering the set of marked points) a pointed stable curve

$$(92) \quad \mathfrak{P}_{m,j}^{(l+2)\text{pt}} := (P_j, \{\sigma_{m,i}^l\}_{(j-1)l < i \leq jl} \cup \{\delta_{j-1}, \delta_j\})$$

(where $\delta_0 := \delta_m$) over k of type $(0, l+2)$. Denote by

$$(93) \quad P_j^{\mathfrak{C}_m^l\text{-log}}$$

the log scheme over $\text{Spec}(k)^{\mathfrak{C}_m^l\text{-log}}$ which is defined as the k -scheme P_j equipped with the log structure pulled-back from the log structure of $C_m^{\mathfrak{C}_m^l\text{-log}}$ via the natural closed immersion $P_j \rightarrow C_m$. Then, we have the commutative square diagram

$$(94) \quad \begin{array}{ccc} P_j^{\mathfrak{C}_m^l\text{-log}} & \longrightarrow & P_j^{\mathfrak{P}_{m,j}^{(l+2)\text{pt}}\text{-log}} \\ \downarrow & & \downarrow \\ \text{Spec}(k)^{\mathfrak{C}_m^l\text{-log}} & \longrightarrow & \text{Spec}(k)^{\mathfrak{P}_{m,j}^{(l+2)\text{pt}}\text{-log}}. \end{array}$$

The following lemma is immediately verified.

Lemma 4.1.1.

The resulting morphism

$$(95) \quad \Phi_{m,j}^l : P_j^{\mathfrak{C}_m^l\text{-log}} \rightarrow P_j^{\mathfrak{P}_{m,j}^{(l+2)\text{pt}}\text{-log}} \times_{k^{\mathfrak{P}_{m,j}^{(l+2)\text{pt}}\text{-log}}} k^{\mathfrak{C}_m^l\text{-log}}$$

over $\text{Spec}(k)^{\mathfrak{C}_m^l\text{-log}}$ is log étale and its underlying morphism of k -schemes coincides with the identity morphism of P_j ($= \mathbb{P}$).

4.2. Let us consider the degeneration of the endomorphism of an elliptic curve given by multiplication by a positive integer l prime to p .

Let m be a positive integer, and let $v : \operatorname{Spec}(k) \rightarrow \overline{\mathfrak{M}}_{1,m}$ be a k -rational point of the moduli stack $\overline{\mathfrak{M}}_{1,m}$ classifying the pointed stable curve \mathfrak{C}_m^1 . The divisor at infinity of $\overline{\mathfrak{M}}_{1,m}$ around v may be locally given as $\prod_{i=1}^m t_i = 0$ for some local functions t_i ($i = 1, \dots, m$). Then, the choice of sections t_i 's determines a morphism $v_{R_1} : \operatorname{Spec}(R_1) \rightarrow \overline{\mathfrak{M}}_{1,m}$, where $R_1 := k[[t_1, \dots, t_m]]$. Denote by

$$(96) \quad \mathfrak{C}_{m,R_1}^1 := (C_{m,R_1}/R_1, \{\sigma_{m,R_1,i}^1\}_{i=1}^m)$$

the pointed stable curve over R_1 of type $(1, m)$ classified by v_{R_1} . (Hence, the spacial fiber is isomorphic to \mathfrak{C}_m^1 and its underlying semistable curve of the generic fiber is smooth.) Denote by $\operatorname{Spec}(R_1)^{\log}$ the log scheme defined as $\operatorname{Spec}(R_1)$ equipped with the log structure associated with the homomorphism $\mathbb{N}^{\oplus m} \rightarrow R_1$ given by assigning $(a_i)_{i=1}^m \mapsto \prod_{i=1}^m t_i^{a_i}$. Then, there exists a canonical isomorphism

$$(97) \quad \eta_1^{\log} : \operatorname{Spec}(R_1)^{\mathfrak{C}_{m,R_1}^1 - \log} \xrightarrow{\sim} \operatorname{Spec}(R_1)^{\log}$$

(whose underlying morphism of schemes coincides with the identity).

Let us write $R_l := k[[t_1^{\frac{1}{l}}, \dots, t_m^{\frac{1}{l}}]]$ (resp., $R_\infty := \varinjlim_{p \nmid n} k[[t_1^{\frac{1}{n}}, \dots, t_m^{\frac{1}{n}}]]$) and equip $\operatorname{Spec}(R_l)$ (resp., $\operatorname{Spec}(R_\infty)$) with a log structure associated with the homomorphism $(\frac{1}{l}\mathbb{N})^{\oplus m} \rightarrow R_l$ (resp., $\varinjlim_{p \nmid n} (\frac{1}{n}\mathbb{N})^{\oplus m} \rightarrow R_\infty$) given by $(a_i)_{i=1}^m \mapsto \prod_{i=1}^m t_i^{a_i}$. Write $\operatorname{Spec}(R_l)^{\log}$ (resp., $\operatorname{Spec}(R_\infty)^{\log}$) for the resulting log scheme and write $\operatorname{Spec}(k)^{\log}$ for the log scheme defined as $\operatorname{Spec}(k)$ equipped with the log structure pulled-back from the log structure of $\operatorname{Spec}(R_\infty)^{\log}$ via the closed immersion $\operatorname{Spec}(k) \rightarrow \operatorname{Spec}(R_\infty)$. Then, the sequence of homomorphisms $R_1 \hookrightarrow R_l \hookrightarrow R_\infty \twoheadrightarrow k$ yield a sequence of morphisms of log schemes

$$(98) \quad \begin{aligned} \operatorname{Spec}(k)^{\log} &\rightarrow \operatorname{Spec}(R_\infty)^{\log} \rightarrow \operatorname{Spec}(R_l)^{\log} \\ &\rightarrow \operatorname{Spec}(R_1)^{\log} \xrightarrow{(\eta_1^{\log})^{-1}} \operatorname{Spec}(R_1)^{\mathfrak{C}_{m,R_1}^1 - \log} \end{aligned}$$

(by which $\operatorname{Spec}(R_\infty)^{\log}$ may be thought of as a universal Kummer étale covering of $\operatorname{Spec}(R)^{\mathfrak{C}_{m,R}^1 - \log}$).

Let us fix an algebraically closed field K together with an inclusion $R_\infty \hookrightarrow K$, which induces a morphism $\operatorname{Spec}(K) \rightarrow \operatorname{Spec}(R_\infty)^{\log}$. Then, let

$$(99) \quad \mathfrak{C}_{m,K}^1 := (C_{m,K}, \{\sigma_{m,K,i}^1\}_{i=1}^m)$$

be the base-change of \mathfrak{C}_{m,R_1}^1 via this morphism. Write

$$(100) \quad C_{m,R_\infty}^{\mathfrak{C}_{m,R_1}^1 - \log^\square} := C_{m,R_1}^{\mathfrak{C}_{m,R_1}^1 - \log^\square} \times_{R_1^{\mathfrak{C}_{m,R_1}^1 - \log}} R_\infty^{\log}.$$

where \square denotes either the present or absence of the prime “ ’ ”. (Hence, we have $C_{m,K} \cong C_{m,R_\infty}^{\mathfrak{C}_{m,R_1}^1 - \log'} \times_{R_\infty^{\log}} K$). The natural morphisms $C_{m,K} \rightarrow C_{m,R_\infty}^{\mathfrak{C}_{m,R_1}^1 - \log'}$

and $C_{m,R_\infty}^{\mathfrak{e}_{m,R_1}^1 - \log} \rightarrow C_{m,R_\infty}^{\mathfrak{e}_{m,R_1}^1 - \log'}$ give rise to a diagram of logarithmic fundamental groups

$$(101) \quad \begin{array}{ccc} & \pi_1^{\text{ket}}(C_{m,R_\infty}^{\mathfrak{e}_{m,R_1}^1 - \log}) & \\ & \downarrow & \\ \pi_1^{\text{et}}(C_{m,K}) & \longrightarrow & \pi_1^{\text{ket}}(C_{m,R_\infty}^{\mathfrak{e}_{m,R_1}^1 - \log'}) \left(\xrightarrow{\sim} \pi_1^{\text{ket}}(C_m^{\mathfrak{e}_m^1 - \log'} \times_{k^{\mathfrak{e}_m^1 - \log}} k^{\log}) \right) \end{array}$$

where all the arrows are determined up to choices of base point, i.e., up to composition with inner automorphism. The vertical arrow in (101) is surjective and the horizontal arrow becomes an isomorphism after taking maximal pro- l quotient $(-)^{(l)}$. Here, $[l]_{C_{m,K}}$ denotes the endomorphism of $C_{m,K}$ determined by multiplication by l , which is an abelian étale covering of degree l^2 . There exist uniquely (up to isomorphism) a log-curve C_{m,l,R_∞}^{\log} over R_∞^{\log} and a Kummer étale covering

$$(102) \quad [l]_{C_{m,R_\infty}^{\log}}^{\log} : C_{m,l,R_\infty}^{\log} \rightarrow C_{m,R_\infty}^{\mathfrak{e}_{m,R_1}^1 - \log}$$

over $\text{Spec}(R_\infty)^{\log}$ corresponding to the abelian covering $[l]_{C_{m,K}}$ via the resulting surjection

$$(103) \quad \pi_1^{\text{ket}}(C_{m,R_\infty}^{\mathfrak{e}_{m,R_1}^1 - \log}) \twoheadrightarrow \pi_1^{\text{ket}}(C_{m,R_\infty}^{\mathfrak{e}_{m,R_1}^1 - \log'})^{(l)} \xrightarrow{\sim} \pi_1^{\text{et}}(C_{m,K})^{(l)}.$$

(Namely, $C_{m,l,R_\infty} \times_{R_\infty} K \cong C_{m,K}$ and the fiber of $[l]_{C_{m,R_\infty}}$ over K coincides with $[l]_{C_{m,K}}$.) One may find a collection of data described as follows:

- a pointed stable curve

$$(104) \quad \mathfrak{e}_{m,l,R_l}^l := (C_{m,l,R_l}/R_l, \{\sigma_{m,l,R_l,i}^l\}_{i=1}^{m \cdot l^2})$$

of type $(1, m \cdot l^2)$ over R_l (which is uniquely determined up to change of ordering in the marked points $\{\sigma_{m,l,R_l,i}^l\}_{i=1}^{m \cdot l^2}$);

- an isomorphism

$$(105) \quad \eta_l^{\log} : \text{Spec}(R_l)^{\log} \xrightarrow{\sim} \text{Spec}(R_l)^{\mathfrak{e}_{m,l,R_l}^l - \log}$$

of log schemes whose underlying morphism of schemes coincides with the identity;

- a commutative diagram

$$(106) \quad \begin{array}{ccccc} C_{m,l,R_\infty}^{\log} & \longrightarrow & C_{m,l,R_l}^{\mathfrak{C}_{m,l,R_l}^l - \log} & \longrightarrow & C_{m,R_1}^{\mathfrak{C}_{m,R_1}^1 - \log} \\ \downarrow & & \downarrow & & \downarrow \\ & & \mathrm{Spec}(R_l)^{\mathfrak{C}_{m,l,R_l}^l - \log} & \longrightarrow & \mathrm{Spec}(R_1)^{\mathfrak{C}_{m,R_1}^1 - \log} \\ & & \downarrow \wr \eta_l^{\log} & & \downarrow \wr \eta_1^{\log} \\ \mathrm{Spec}(R_\infty)^{\log} & \longrightarrow & \mathrm{Spec}(R_l)^{\log} & \longrightarrow & \mathrm{Spec}(R_1)^{\log}, \end{array}$$

where both the left-hand rectangle and the right-hand lower square are cartesian, the two lower horizontal arrows are obtained in (98), and all the upper vertical arrows except η_1^{\log} and η_l^{\log} are the structure morphisms of pointed stable curves.

The fact that the left-hand rectangle is cartesian gives an isomorphism

$$(107) \quad C_{m,l,R_\infty}^{\log} \xrightarrow{\sim} C_{m,l,R_l}^{\mathfrak{C}_{m,l,R_l}^l - \log} \times_{\mathrm{Spec}(R_l)^{\log}} \mathrm{Spec}(R_\infty)^{\log}$$

over $\mathrm{Spec}(R_\infty)^{\log}$. Also, one verifies that the special fiber of \mathfrak{C}_{m,l,R_l}^l is isomorphic to $\mathfrak{C}_{m,l}^l := (C_{m,l}/k, \{\sigma_{m,l,i}^l\}_{i=1}^{m \cdot l^2})$. Denote by

$$(108) \quad [l]_{C_m}^{\log} : C_{m,l,k}^{\mathfrak{C}_{m,l,k}^l - \log} \times_{k^{\mathfrak{C}_{m,l,k}^l - \log}} k^{\log} \rightarrow C_m^{\mathfrak{C}_m^1 - \log} \times_{k^{\mathfrak{C}_m^1 - \log}} k^{\log}$$

the base-change of $[l]_{C_m,R_\infty}^{\log}$ over k^{\log} under the isomorphism (107). The image of each $P_j \subseteq C_{m,l}$ ($j = 1, \dots, m \cdot l$) via $[l]_{C_m}$ is contained in $P_{s_j} \subseteq C_m$ for some $s_j \in \{1, \dots, m\}$. Thus, we obtain its restriction of the following form

$$(109) \quad [l]_{C_m}^{\log} |_{P_j} : P_j^{\mathfrak{C}_{m,l}^l - \log} \times_{k^{\mathfrak{C}_{m,l}^l - \log}} k^{\log} \rightarrow P_{s_j}^{\mathfrak{C}_m^1 - \log} \times_{k^{\mathfrak{C}_m^1 - \log}} k^{\log}.$$

By a straightforward argument, one verifies the following lemma.

Lemma 4.2.1.

The square diagram

$$(110) \quad \begin{array}{ccc} P_j^{\mathfrak{C}_{m,l}^l - \log} \times_{k^{\mathfrak{C}_{m,l}^l - \log}} k^{\log} & \xrightarrow{[l]_{C_m}^{\log} |_{P_j}} & P_{s_j}^{\mathfrak{C}_m^1 - \log} \times_{k^{\mathfrak{C}_m^1 - \log}} k^{\log} \\ \Phi_{m,l,j}^l \times \mathrm{id}_{k^{\log}} \downarrow & & \downarrow \Phi_{m,s_j}^1 \times \mathrm{id}_{k^{\log}} \\ P_j^{\mathfrak{P}_{m,j}^{(l+2)\mathrm{pt}} - \log} \times_k k^{\log} & \xrightarrow{\pi^{\log}} & P_{s_j}^{\mathfrak{P}_{m,j}^{3\mathrm{pt}} - \log} \times_k k^{\log} \end{array}$$

(cf. (79) for the definition of π^{\log}) is commutative and cartesian.

By applying the above lemma, one may prove the following proposition.

Proposition 4.2.2.

Write $\mathfrak{E}_{m,l,K}^l$ for the base-change of \mathfrak{E}_{m,l,R_1}^l via the morphism $\mathrm{Spec}(K) \rightarrow \mathrm{Spec}(R_\infty)$. In particular, we obtain an abelian covering $(\mathfrak{E}_{m,l,K}^l, [l]_{C_{m,K}})$ of $\mathfrak{E}_{m,K}^1$ over K defined as the base-change of $[l]_{C_{m,R_\infty}}$. Then, the pull-back $[l]_{C_{m,K}}^*(\mathcal{F})^\heartsuit$ of any GL_n -do'per \mathcal{F}^\heartsuit on $\mathfrak{E}_{m,K}^1$ is \otimes -ordinary.

Proof. By Proposition 2.5.2 (i), $\mathfrak{Op}_{\mathrm{GL}_n,1,m}^{\mathrm{Zzz...}} \times_{\mathfrak{M}_{1,m,v_{R_1}}} \mathrm{Spec}(R_1)$ is isomorphic to a disjoint union of finite copies of $\mathrm{Spec}(R_1)$. Hence, there exists a morphism $\tilde{v}_{R_1} : \mathrm{Spec}(R_1) \rightarrow \mathfrak{Op}_{\mathrm{GL}_n,1,m}^{\mathrm{Zzz...}}$ whose composite with $\mathrm{Spec}(K) \rightarrow \mathrm{Spec}(R_1)$ coincides with the classifying morphism of \mathcal{F}^\heartsuit . If $\mathcal{F}_{R_1}^\heartsuit$ denotes the GL_n -do'per on \mathfrak{E}_{m,R_1}^1 , then its pull-back via $C_{m,R_\infty}^{\mathfrak{E}_{m,R_1}^1-\log} \rightarrow C_{m,R_1}^{\mathfrak{E}_{m,R_1}^1-\log}$ (resp., the composite $C_{m,l,R_\infty}^{\log} \xrightarrow{[l]_{C_{m,R_\infty}}^{\log}} C_{m,R_\infty}^{\mathfrak{E}_{m,R_1}^1-\log} \rightarrow C_{m,R_1}^{\mathfrak{E}_{m,R_1}^1-\log}$) forms a GL_n -do'per $\mathcal{F}_{m,R_\infty^{\log}}^\heartsuit$ on the log-curve $C_{m,R_\infty}^{\mathfrak{E}_{m,R_1}^1-\log}/\mathrm{Spec}(R_\infty)^{\log}$ (resp., a GL_n -do'per $\mathcal{F}_{m,l,R_\infty^{\log}}^\heartsuit$ on the log-curve $C_{m,l,R_\infty}^{\log}/\mathrm{Spec}(R_\infty)^{\log}$). Denote by $\mathcal{F}_{m,k^{\log}}^\heartsuit$ (resp., $\mathcal{F}_{m,l,k^{\log}}^\heartsuit$) the GL_n -do'per on $C_m^{\mathfrak{E}_m^1-\log} \times_{k^{\mathfrak{E}_m^1-\log}} k^{\log}/k^{\log}$ (resp., $C_{m,l}^{\mathfrak{E}_{m,l}^1-\log} \times_{k^{\mathfrak{E}_{m,l}^1-\log}} k^{\log}/k^{\log}$) defined as the restriction of $\mathcal{F}_{m,R_\infty^{\log}}^\heartsuit$ (resp., $\mathcal{F}_{m,l,R_\infty^{\log}}^\heartsuit$) to the special fiber. Since the generic fiber of $\mathcal{F}_{m,l,R_\infty^{\log}}^\heartsuit$ is isomorphic to $[l]_{C_{m,K}}^*(\mathcal{F})^\heartsuit$, it suffices (by Corollary 2.5.3) to verify the \otimes -ordinariness of $\mathcal{F}_{m,l,k^{\log}}^\heartsuit$. To this end, we first observe from Lemma 4.1.1 that for each $j \in \{1, \dots, m \cdot l\}$, the restriction $\mathcal{F}_{m,k^{\log}}^\heartsuit|_{P_{s_j}^\heartsuit}$ of $\mathcal{F}_{m,k^{\log}}^\heartsuit$ to $P_{s_j}^{\mathfrak{E}_m^1-\log} \times_{k^{\mathfrak{E}_m^1-\log}} k^{\log}/k^{\log}$ may be identified with a GL_n -do'per on $P_{s_j}^{\mathfrak{P}_{m,j}^{\mathrm{3pt}}-\log} \times k^{\log}/k^{\log}$ via $\Phi_{m,s_j}^1 \times \mathrm{id}_{k^{\log}}$ (cf. (95)). By Proposition 3.3.1, the pull-back $\pi^*(\mathcal{F}_{m,k^{\log}}^\heartsuit|_{P_{s_j}^\heartsuit})^\heartsuit$ via π^{\log} of this GL_n -do'per is \otimes -ordinary. By Lemma 4.1.1 again, this pull-back may be identified with a GL_n -do'per on $P_j^{\mathfrak{E}_{m,l}^1-\log} \times_{k^{\mathfrak{E}_{m,l}^1-\log}} k^{\log}/k^{\log}$, which is isomorphic to the restriction of $\mathcal{F}_{m,l,k^{\log}}^\heartsuit$. Thus, it follows from Proposition 2.6.1 that $\mathcal{F}_{m,l,k^{\log}}^\heartsuit$ is \otimes -ordinary. This completes the proof of Proposition 4.2.2. \square

4.3. The proof of Theorem B. This subsection is devoted to prove Theorem B. Let G be a finite abelian group with $p \nmid \#(G)$.

Definition 4.3.1.

Let S be a scheme over k and $\mathfrak{X} := (X/S, \{\sigma_i\}_i)$ a pointed proper smooth curve over S . We shall say that \mathfrak{X} is $\otimes_{n,G}$ -ordinary if for any \otimes -ordinary GL_n -do'per \mathcal{F}^\heartsuit on \mathfrak{X} and any abelian covering (\mathfrak{Y}, w, v) of \mathfrak{X} with Galois group $\mathrm{Gal}(\mathfrak{Y}/\mathfrak{X}) \cong G$ such that v is étale, the pull-back $w^*(\mathcal{F})^\heartsuit$ is \otimes -ordinary.

Denote by

$$(111) \quad {}^{\circledast}\mathfrak{M}_{g,r,n,G}$$

the substack of $\mathfrak{M}_{g,r}$ classifying $\circledast_{n,G}$ -ordinary pointed proper smooth curves. The proof of Theorem B reduces to proving the assertion that ${}^{\circledast}\mathfrak{M}_{g,r,n,G}$ is a dense open substack of $\mathfrak{M}_{g,r}$. Thus, it suffices to prove the following Lemma 4.3.2 and Lemma 4.3.3.

Lemma 4.3.2.

${}^{\circledast}\mathfrak{M}_{g,r,n,G}$ is an open substack of $\mathfrak{M}_{g,r}$

Proof. Denote by $\mathfrak{N}_{g,r,n,G}$ the étale sheaf on $\mathfrak{M}_{g,r}$ associated with the presheaf which, to any étale scheme S over $\mathfrak{M}_{g,r}$ classifying a pointed stable curve \mathfrak{X} , assigns the set of isomorphism classes of étale coverings $(\mathfrak{Y}, w, \text{id}_S)$ of \mathfrak{X} over S with Galois group $\text{Gal}(\mathfrak{Y}/\mathfrak{X}) \cong G$. Then, $\mathfrak{N}_{g,r,n,G}$ may be represented by a relative finite scheme over $\mathfrak{M}_{g,r}$. The assignment $(\mathfrak{Y}, w, \text{id}_S) \mapsto w^*(\mathcal{F})^\heartsuit$ defines a morphism

$$(112) \quad \xi : \mathfrak{N}_{g,r,n,G} \times_{\overline{\mathfrak{M}}_{g,r}} \mathfrak{Op}_{\text{GL}_n, g, r}^{\text{Zzz}\dots} \rightarrow \mathfrak{Op}_{\text{GL}_n, (g-1)\cdot\sharp(G)+1, r\cdot\sharp(G)}^{\text{Zzz}\dots}$$

Recall from Proposition 2.5.1 (i) that ${}^{\circledast}\mathfrak{Op}_{\text{GL}_n, (g-1)\cdot\sharp(G)+1, r\cdot\sharp(G)}^{\text{Zzz}\dots}$ is an open substack of $\mathfrak{Op}_{\text{GL}_n, (g-1)\cdot\sharp(G)+1, r\cdot\sharp(G)}^{\text{Zzz}\dots}$, and hence,

$$(113) \quad (\mathfrak{N}_{g,r,n,G} \times_{\overline{\mathfrak{M}}_{g,r}} \mathfrak{Op}_{\text{GL}_n, g, r}^{\text{Zzz}\dots}) \setminus \xi^{-1}({}^{\circledast}\mathfrak{Op}_{\text{GL}_n, (g-1)\cdot\sharp(G)+1, r\cdot\sharp(G)}^{\text{Zzz}\dots})$$

is a closed substack of $\mathfrak{N}_{g,r,n,G} \times_{\overline{\mathfrak{M}}_{g,r}} \mathfrak{Op}_{\text{GL}_n, g, r}^{\text{Zzz}\dots}$. On the other hand, the natural projection

$$(114) \quad \mathfrak{N}_{g,r,n,G} \times_{\overline{\mathfrak{M}}_{g,r}} \mathfrak{Op}_{\text{GL}_n, g, r}^{\text{Zzz}\dots} \rightarrow \mathfrak{M}_{g,r}$$

is proper since both $\mathfrak{N}_{g,r,n,G}/\mathfrak{M}_{g,r}$ and $\mathfrak{Op}_{\text{GL}_n, g, r}^{\text{Zzz}\dots}/\overline{\mathfrak{M}}_{g,r}$ are proper (cf. [23], Theorem C). Hence, ${}^{\circledast}\mathfrak{M}_{g,r,n,G}$, which coincides with the complement of the image of (113) via (114), is open in $\mathfrak{M}_{g,r}$, as desired. \square

Lemma 4.3.3.

Any geometric generic point $\text{Spec}(L) \rightarrow \mathfrak{M}_{g,r}$ (where L denotes an algebraically closed field over k) of $\mathfrak{M}_{g,r}$ lies in ${}^{\circledast}\mathfrak{M}_{g,r,n,G}$.

Proof. Denote by $\mathfrak{C}_L := (C_L/L, \{\sigma_{L,i}\}_{i=1}^r)$ the pointed proper smooth curve classified by the point $\text{Spec}(L) \rightarrow \mathfrak{M}_{g,r}$. Let \mathcal{F}^\heartsuit be a GL_n -do'per on \mathfrak{C}_L and (\mathfrak{C}'_L, w_L) (where $\mathfrak{C}'_L := (C'_L/L, \{\sigma'_{L,i}\}_i)$) an abelian covering of \mathfrak{C}_L over L with Galois group $\text{Gal}(\mathfrak{C}'_L/\mathfrak{C}_L) \cong G$. What we have to prove is that $w_L^*(\mathcal{F})^\heartsuit$ is \circledast -ordinary. (Here, since there is no nontrivial étale covering of the projective line over L , we only consider the case where $g \geq 1$.)

First, we shall consider the case where $g = 1$. We may assume, without loss of generality, that $L = K$ and $r = m$ as in § 4.2 (hence, \mathfrak{C}_K is isomorphic to $\mathfrak{C}_{m,K}^1$).

One may find an étale covering $w'_K : C_{m,K} \rightarrow C'$ such that $w_K \circ w'_K = [l]_{C_{m,K}}$ for some positive integer l prime to p . By Theorem A, the proof reduces to proving that $[l]_{C_{m,K}}^*(\mathcal{F})^\heartsuit$ is \otimes -ordinary. Hence, the assertion follows from Proposition 4.2.2. (Moreover, by combining with Theorem A, it holds that \mathfrak{C}_K is $\otimes_{n,G'}$ -ordinary for any quotient group G' of G . Thus, it follows from Lemma 4.3.2 that any general pointed proper smooth curve is $\otimes_{n,G'}$ -ordinary.)

Next, we consider the case $g \geq 2$. Let $\mathfrak{C}_k := (C_k/k, \{\sigma_{k,i}\}_{i=1}^r)$ be a pointed stable curve of type (g, r) over k obtained (from the argument in § 2.6) by gluing together g pointed elliptic curves $\mathfrak{C}_{k,j} := (C_{k,j}/k, \{\sigma_{k,j,i}\}_i)$ (by means of a certain clutching data) in such a way that the dual graph of \mathfrak{C}_k is tree. By the above discussion, one may find such a curve \mathfrak{C}_k satisfying furthermore that each $\mathfrak{C}_{k,j}$ is $\otimes_{n,G'}$ -ordinary for any quotient group G' of G . After possibly replacing L with its extension field, there exist an inclusion $R \hookrightarrow L$ (over k), where $R := k[[t_1, \dots, t_m]]$ (for some m), and a pointed stable curve

$$(115) \quad \mathfrak{C}_R := (C_R/R, \{\sigma_{R,i}\}_{i=1}^r)$$

over R of type (g, r) satisfying the following properties:

- If s denotes the closed point of $\mathrm{Spec}(R)$, then the fiber of \mathfrak{C}_R over s is isomorphic to \mathfrak{C}_k ;
- If η denotes the geometric generic point $\mathrm{Spec}(L) \rightarrow \mathrm{Spec}(R)$ corresponding to the inclusion $R \hookrightarrow L$, then fiber of \mathfrak{C}_R over η is isomorphic to \mathfrak{C}_L .

One verifies from the definition of \mathfrak{C}_R that there exists a Galois covering $w_R : C'_R \rightarrow C_R$ over R whose base-change via η coincides with $w_L : C'_L \rightarrow C_L$. (Indeed, the natural map $\pi_1^{\mathrm{ét}}(C_L) \rightarrow \pi_1^{\mathrm{ét}}(C_R)$ between étale fundamental groups is bijective after taking maximal abelian pro-prime-to- p quotient.) One may find a set of marked points $\{\sigma'_{R,i}\}_i$ of C'_R for which $\mathfrak{C}'_R := (C'_R/R, \{\sigma'_{R,i}\}_i)$ forms a pointed stable curve over R and (\mathfrak{C}'_R, w_R) extends to an étale covering (\mathfrak{C}'_R, w_R) of \mathfrak{C}_R . The special fiber $\mathfrak{C}'_k := (C'_R \times_{R,s} k, \{\sigma'_{R,i} \times \mathrm{id}_k\}_i)$ of \mathfrak{C}'_R may be obtained by gluing together some pointed proper smooth curves $\mathfrak{C}'_{k,j}$ ($j = 1, 2, \dots$). For each j , the fiber (\mathfrak{C}'_k, w_k) of (\mathfrak{C}'_R, w_R) over s restricts to an abelian étale covering $(\mathfrak{C}'_{k,j}, w_{k,j})$ of \mathfrak{C}_{k,s_j} with Galois group G_j (where $s_j \in \{1, \dots, g\}$ and G_j denotes some quotient group of G). Here, observe that since $\mathfrak{Op}_{\mathrm{GL}_n, g, r}^{\mathrm{Zzz...}}$ is finite and generically étale over $\overline{\mathfrak{M}}_{g,r}$, there exists a GL_n -do'per \mathcal{F}_R^\heartsuit on \mathfrak{C}_R which restricts to \mathcal{F}^\heartsuit . By the condition on each $\mathfrak{C}_{k,j}$ assumed earlier, the restrictions $\mathcal{F}|_{\mathfrak{C}_{k,s_j}}^\heartsuit$ of \mathcal{F}^\heartsuit to \mathfrak{C}_{k,s_j} , as well as its pull-back via $w_{k,j}$, is \otimes -ordinary. Hence, it follows from Proposition 2.6.1 that $w_k^*(\mathcal{F})^\heartsuit$ is \otimes -ordinary. Finally, by Corollary 2.5.3, $w_L^*(\mathcal{F})^\heartsuit$ is \otimes -ordinary, and hence, we complete the proof of Lemma 4.3.3. \square

In particular, we have completed the proof of Theorem B.

5. ORDINARINESS FOR A SEMISIMPLE LIE ALGEBRA \mathfrak{g}

As shown in Proposition 2.5.1 (or Proposition 2.5.2, (i)), the ordinariness, as well as the \circledast -ordinariness, of GL_n -do'pers is closely related to the unramifiedness of the moduli stack $\mathfrak{Op}_{\mathfrak{sl}_n, g, r}^{\mathrm{Zzz}\dots}$ at their classifying morphisms. In a similar vein, one may define the ordinariness of dormant \mathfrak{g} -opers for a certain class of semisimple Lie algebras \mathfrak{g} and consider the sorts of problems discussed before for dormant \mathfrak{g} -opers.

5.1. For example, let \mathfrak{g} be a semisimple Lie algebra over k whose rank is, in a certain sense (cf. the conditions $(\mathrm{Char})_0$ and $(\mathrm{Char})_p^{\mathrm{W}}$ described in [23], § 2.1), sufficiently small relative to the characteristic p of k . Once we fix a certain collection of data concerning such a \mathfrak{g} (e.g., a choice of an \mathfrak{sl}_2 -triple $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$ and a pinning that are compatible in a certain sense (cf. [23], § 2.1)), it makes sense to speak about the notion of a dormant \mathfrak{g} -oper on a pointed stable curve. In the following, when we mention a semisimple \mathfrak{g} , we always assume that such a collection of data have been chosen. Roughly speaking, a dormant \mathfrak{g} -oper on a pointed stable curve $\mathfrak{X} := (X/S, \{\sigma_i\}_i)$ is a pair $\mathcal{E}^\spadesuit := (\mathcal{E}, \nabla_{\mathcal{E}})$ consisting of an $\mathrm{Aut}^0(\mathfrak{g})$ -torsor \mathcal{E} over X (where $\mathrm{Aut}^0(\mathfrak{g})$ denotes the identity component of the group of Lie algebra automorphisms of \mathfrak{g}) and an (integrable) S -log connection $\nabla_{\mathcal{E}}$ on \mathcal{E} satisfying certain conditions, including vanishing p -curvature. We refer to [23], Definition 2.2.1, and Definition 3.6.1, for the precise definition of a dormant \mathfrak{g} -oper. As in the previous discussion, we shall abbreviate a dormant \mathfrak{g} -oper to a **\mathfrak{g} -do'per**. If, moreover, (\mathfrak{Y}, w, v) is an étale covering of \mathfrak{X} , then by pulling-back the data in \mathcal{E}^\spadesuit via w , one may obtain a \mathfrak{g} -do'per $w^*(\mathcal{E})^\spadesuit$ on \mathfrak{Y} , which we refer to as the **pull-back of \mathcal{E}^\spadesuit** .

Denote by $\mathfrak{Op}_{\mathfrak{g}, g, r}^{\mathrm{Zzz}\dots}$ the moduli stack classifying pairs $(\mathfrak{X}, \mathcal{E}^\spadesuit)$ consisting of a pointed stable curve of type (g, r) over k and a \mathfrak{g} -do'per \mathcal{E}^\spadesuit on it. By forgetting the data of a \mathfrak{g} -do'per, we have a morphism $\mathfrak{Op}_{\mathfrak{g}, g, r}^{\mathrm{Zzz}\dots} \rightarrow \overline{\mathfrak{M}}_{g, r}$. Denote by

$$(116) \quad {}^\circ\mathfrak{Op}_{\mathfrak{g}, g, r}^{\mathrm{Zzz}\dots}$$

the unramified (equivalently, étale, due to [23], Corollary 5.12.2) locus of $\mathfrak{Op}_{\mathfrak{g}, g, r}^{\mathrm{Zzz}\dots}$ over $\overline{\mathfrak{M}}_{g, r}$; it forms a possibly empty open substack of $\mathfrak{Op}_{\mathfrak{g}, g, r}^{\mathrm{Zzz}\dots}$.

Definition 5.1.1.

Let \mathfrak{X} be a pointed stable curve over a k -scheme S of type (g, r) . We shall say that a \mathfrak{g} -do'per \mathcal{E}^\spadesuit on \mathfrak{X} is **$\circledast_{\mathfrak{g}}$ -ordinary** if its classifying map $S \rightarrow \mathfrak{Op}_{\mathfrak{g}, g, r}^{\mathrm{Zzz}\dots}$ factors through the open immersion ${}^\circ\mathfrak{Op}_{\mathfrak{g}, g, r}^{\mathrm{Zzz}\dots} \hookrightarrow \mathfrak{Op}_{\mathfrak{g}, g, r}^{\mathrm{Zzz}\dots}$.

The notion of $\circledast_{\mathfrak{sl}_n}$ -ordinariness is, via the isomorphism (50), equivalent to the \circledast -ordinariness of GL_n -do'pers discussed so far. More precisely, let \mathcal{F}^\heartsuit

be a GL_n -do'per and denote by \mathcal{E}^\spadesuit the \mathfrak{sl}_n -do'per associated with \mathcal{F}^\heartsuit . Then, it follows from the equivalence (i) \Leftrightarrow (iii) in Proposition 2.5.1 that \mathcal{F}^\heartsuit is \otimes -ordinary if and only if \mathcal{E}^\spadesuit is $\otimes_{\mathfrak{sl}_n}$ -ordinary. The result described in Theorem B generalizes to some \mathfrak{g} . In fact, we shall consider simple Lie algebras \mathfrak{g} introduced in the following definition.

Definition 5.1.2.

We shall say that \mathfrak{g} is **admissibly of classical type A** if \mathfrak{g} is isomorphic to either \mathfrak{sl}_n (for $n < p$), \mathfrak{so}_{2n+1} (for $n < \frac{p-1}{2}$), or \mathfrak{sp}_{2n} (for $n < \frac{p}{2}$).

If \mathfrak{g} is admissibly of classical type A , then the locus ${}^\odot\mathfrak{Op}_{\mathfrak{g},g,r}^{\mathrm{Zzz}\dots}$ is dense and open in each irreducible component of $\mathfrak{Op}_{\mathfrak{g},g,r}^{\mathrm{Zzz}\dots}$ that dominates $\overline{\mathfrak{M}}_{g,r}$ (cf. [23], Theorem F), as in the case of \mathfrak{sl}_n (cf. Proposition 2.5.2 (i)). Moreover, note that one may find an embedding $\mathfrak{g} \rightarrow \mathfrak{sl}_m$ (for some $m \leq p-1$) that preserves the respective pinnings and \mathfrak{sl}_2 -triples (cf. [23], § 7.9). This embedding yields a closed immersion

$$(117) \quad \iota_{\mathfrak{g} \rightarrow \mathfrak{sl}_m} : \mathfrak{Op}_{\mathfrak{g},g,r}^{\mathrm{Zzz}\dots} \rightarrow \mathfrak{Op}_{\mathfrak{sl}_m,g,r}^{\mathrm{Zzz}\dots}$$

over $\overline{\mathfrak{M}}_{g,r}$.

Proposition 5.1.3.

Let \mathfrak{X} be a pointed stable curve over a k -scheme of type (g, r) and \mathcal{E}^\spadesuit a \mathfrak{g} -do'per on \mathfrak{X} . Denote by $\mathcal{E}_{\mathfrak{sl}_m}^\spadesuit$ the \mathfrak{sl}_m -do'per on \mathfrak{X} classified by the composite $\iota_{\mathfrak{g} \rightarrow \mathfrak{sl}_m} \circ s : S \rightarrow \mathfrak{Op}_{\mathfrak{sl}_m,g,r}^{\mathrm{Zzz}\dots}$, where s denotes the classifying morphism $s : S \rightarrow \mathfrak{Op}_{\mathfrak{g},g,r}^{\mathrm{Zzz}\dots}$ of \mathcal{E}^\spadesuit . Suppose that $\mathcal{E}_{\mathfrak{sl}_m}^\spadesuit$ is $\otimes_{\mathfrak{sl}_m}$ -ordinary. Then, \mathcal{E}^\spadesuit is $\otimes_{\mathfrak{g}}$ -ordinary.

Proof. The assertion follows from the observation that $\mathfrak{Op}_{\mathfrak{g},g,r}^{\mathrm{Zzz}\dots}$ is unramified (relative to $\overline{\mathfrak{M}}_{g,r}$) at the point s since $\iota_{\mathfrak{g} \rightarrow \mathfrak{sl}_m}$ is a closed immersion and $\mathfrak{Op}_{\mathfrak{sl}_m,g,r}^{\mathrm{Zzz}\dots}$ is unramified at the point $s \circ \iota_{\mathfrak{g} \rightarrow \mathfrak{sl}_m}$. \square

By means of the above proposition, we have the following generalization of Theorem B.

Theorem 5.1.4.

Let \mathfrak{g} be a semisimple Lie algebra over k which is admissibly of classical type A , \mathfrak{X} a pointed proper smooth curve of type (g, r) over k , $(\mathfrak{Y}, w, \mathrm{id}_k)$ an abelian covering of \mathfrak{X} over k , and \mathcal{E}^\spadesuit a \mathfrak{g} -do'per on \mathfrak{X} . Suppose the following conditions:

- (i) \mathcal{E}^\spadesuit is $\otimes_{\mathfrak{g}}$ -ordinary.
- (ii) $(\mathfrak{Y}, w, \mathrm{id}_k)$ is abelian with $p \nmid \#(\mathrm{Gal}(\mathfrak{Y}/\mathfrak{X}))$;
- (iii) \mathfrak{X} is general in the moduli stack $\mathfrak{M}_{g,r}$;

Then, the pull-back $w^*(\mathcal{E})^\spadesuit$ is $\otimes_{\mathfrak{g}}$ -ordinary.

Proof. By Theorem B (and the discussion following Definition 5.1.1), the pull-back (via w) of the \mathfrak{sl}_m -do'per $\mathcal{E}_{\mathfrak{sl}_m}^\bullet$ associated (via $\iota_{\mathfrak{g} \rightarrow \mathfrak{sl}_m}$) with \mathcal{E}^\bullet is $\otimes_{\mathfrak{sl}_m}$ -ordinary. But, since this \mathfrak{sl}_m -do'per on \mathfrak{Y} is isomorphic to the \mathfrak{sl}_m -do'per associated (via $\iota_{\mathfrak{g} \rightarrow \mathfrak{sl}_m}$) with the \mathfrak{g} -do'per $w^*(\mathcal{E})^\bullet$, Proposition 5.1.3 implies that $w^*(\mathcal{E})^\bullet$ is $\otimes_{\mathfrak{g}}$ -ordinary. \square

5.2. We shall conclude the paper with the study of a relationship between $\otimes_{\mathfrak{sl}_2}$ -ordinariness and $\otimes_{\mathfrak{g}}$ -ordinariness.

Definition 5.2.1.

We shall say that a pointed stable curve \mathfrak{X} over a k -scheme S is $\otimes_{\mathfrak{g}}$ -**ordinary** if for any S -scheme T , any \mathfrak{g} -do'per on the base-change of \mathfrak{X} over T is $\otimes_{\mathfrak{g}}$ -ordinary.

Denote by ${}^*\overline{\mathfrak{M}}_{\mathfrak{g},g,r}$ the substack of $\overline{\mathfrak{M}}_{\mathfrak{g},g,r}$ classifying $\otimes_{\mathfrak{g}}$ -ordinary curves. One verifies that ${}^*\overline{\mathfrak{M}}_{\mathfrak{g},g,r}$ coincides with the complement of the image, via the forgetting map $\mathfrak{Op}_{\mathfrak{g},g,r}^{\text{Zzz}\dots} \rightarrow \overline{\mathfrak{M}}_{\mathfrak{g},g,r}$ (which is proper), of the complement of ${}^*\mathfrak{Op}_{\mathfrak{g},g,r}^{\text{Zzz}\dots}$ in $\mathfrak{Op}_{\mathfrak{g},g,r}^{\text{Zzz}\dots}$. In particular, ${}^*\overline{\mathfrak{M}}_{\mathfrak{g},g,r}$ is a possibly empty open substack of $\overline{\mathfrak{M}}_{\mathfrak{g},g,r}$. Then, one may prove the following proposition.

Theorem 5.2.2.

${}^*\overline{\mathfrak{M}}_{\mathfrak{g},g,r}$ is an open substack of ${}^*\overline{\mathfrak{M}}_{\mathfrak{sl}_2,g,r}$. If, moreover, $r = 0$, then ${}^*\overline{\mathfrak{M}}_{\mathfrak{g},g,0}$ (and hence, ${}^*\overline{\mathfrak{M}}_{\mathfrak{sl}_2,g,0}$) is a dense open substack of $\overline{\mathfrak{M}}_{\mathfrak{g},0}$.

Proof. The former assertion follows from the fact that the image of $\mathfrak{Op}_{\mathfrak{sl}_2,g,r}^{\text{Zzz}\dots} \setminus {}^*\mathfrak{Op}_{\mathfrak{sl}_2,g,r}^{\text{Zzz}\dots}$ via the closed immersion $\mathfrak{Op}_{\mathfrak{sl}_2,g,r}^{\text{Zzz}\dots} \rightarrow \mathfrak{Op}_{\mathfrak{g},g,r}^{\text{Zzz}\dots}$ discussed in [23], (134) and Corollary 2.7.6 (i), is contained in $\mathfrak{Op}_{\mathfrak{g},g,r}^{\text{Zzz}\dots} \setminus {}^*\mathfrak{Op}_{\mathfrak{g},g,r}^{\text{Zzz}\dots}$. The latter assertion follows from [23], Theorem C (ii) (and the former assertion). \square

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